Heat Transfer

Conduction heat transfer

Course Topics

1. Heat Transfer Mechanisms.

2. Heat Conduction in Plane Walls, Cylinders and Spheres as well as Multilayered Plane Walls, Cylinders and Spheres.

3. Heat Transfer from Finned Surfaces.

4. Heat Conduction Equation in a Large Plane Wall, Long Cylinder and Sphere.

5. Lumped System Analysis and Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with Spatial Effects.

6. Thermal Radiation, the View Factor and View Factor Relations.

7. Radiation Heat Transfer: Diffuse, Gray Surfaces, Radiosity.

8. Radiation Shields and the Radiation Effect.

Course Description:

This course is an introduction to the principal concepts of heat transfer methods. Heat transfer occurs when the temperatures of objects are not equal to each other and refers to how this difference is changed to an equilibrium state. This course focuses on two different mechanisms of heat transfer; conduction, and radiation.

Recommended Textbook(s):

1. J.P. Holman, "Heat Transfer", 9th Edition, 2013.

2. Yunus A. Cengel, "Heat Transfer, A Practical Approach", 2nd Edition, 2012.

3. F.P. Incropera, D.P. Dewitt, "Fundamentals of Heat and Mass Transfer", 2011.

Lab Topics:

1. Linear Heat Conduction,

2. Radial Heat Conduction,

3. Thermal Conductivity of Liquids,

4. Conduction Heat Transfer from finned surface,

5. Heat Transfer by Radiation.

Chapter one

Basics of Heat Transfer

1.1 Definition

The heat can be defined as the form of energy that can be transferred from one system to another as a result of temperature difference. The science that deals with the determination of the rates of such energy transfers is *heat transfer*.

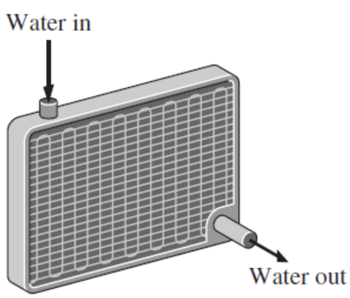
The temperature difference is the driving force for heat transfer, just as the voltage difference is the driving force for electric current flow and pressure difference is the driving force for fluid flow.

1.2 Applications of Heat Transfer

The human body is constantly rejecting heat to its surroundings, and human comfort is closely tied to the rate of this heat rejection. We try to control this heat transfer rate by adjusting our clothing to the environmental conditions.

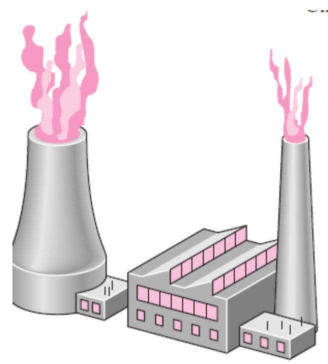
Many ordinary household appliances are designed, in whole or in part, by using the principles of heat transfer such as the heating and air-conditioning system, the refrigerator and freezer, the water heater, the iron, the computer, and the TV. Of course, energy-efficient homes are designed on the basis of minimizing heat loss in winter and heat gain in summer.

Other applications are car radiators, solar collectors, various components of power plants, and even spacecraft. The optimal insulation thickness in the walls and roofs of the houses, on hot water or steam pipes, or on water heaters is again determined on the basis of a heat transfer analysis with economic consideration.



Car radiators


Refrigeration systems


Power plants

1.3 Engineering Heat Transfer

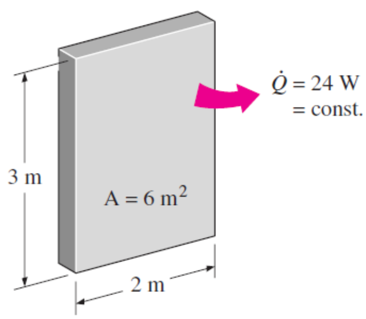
Heat transfer equipment such as heat exchangers, boilers, condensers, radiators, heaters, furnaces, refrigerators, and solar collectors are designed primarily on the basis of heat transfer analysis.

The heat transfer problems encountered in practice can be considered in two groups: (1) rating and (2) sizing problems.

1.4 Forms of the heat

The term heat and the associated phrases such as *heat flow*, *heat addition*, *heat rejection*, *heat absorption*, *heat gain*, *heat loss*, *heat storage*, *heat generation*, *electrical heating*, *latent heat*, *body heat*, and *heat source* are in common use today.

The amount of heat transferred during the process is denoted by (Q) which measured by J. The amount of heat transferred per unit time is called *heat transfer rate*, and is denoted by ($\dot{Q}$) and measured by J/s or W. The rate of heat transfer per unit area normal to the direction of heat transfer is called *heat flux*, which is referred by ($\dot{q}$), and the average heat flux is expressed as $\dot{Q}$ /A, and measured by W/m^2.



$$\dot{q} = \frac{\dot{Q}}{A} = \frac{24\ \text{W}}{6\ \text{m}^2} = 4\ \text{W/m}^2$$

1.5 Heat Transfer Mechanism

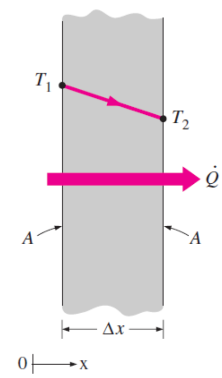
Heat can be transferred in three different modes: *conduction*, *convection*, and *radiation*. All modes of heat transfer require a temperature difference, and moves from the high-temperature medium to the lower-temperature medium.

1.5.1 Conduction Heat Transfer

Conduction is *the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles*. Conduction can take place in solids, liquids, or gases. The rate of heat conduction through a medium depends on the geometry of the medium, its thickness, and the material of the medium, as well as the temperature difference across the medium.

Wrapping a hot water tank with glass wool (an insulating material) reduces the rate of heat loss from the tank. The thicker the insulation, the smaller the heat loss. The hot water tank will lose heat at a higher rate when the temperature of the ambient is lowered. Further, the larger the tank, the larger the surface area and thus the higher rate of heat loss.

The rate of heat conduction through a wall is proportional to the temperature difference across the wall and the heat transfer area, but is inversely proportional to the wall thickness.



$$\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{\Delta x} = -kA\frac{\Delta T}{\Delta x} \qquad \text{(W)} \qquad (1.1)$$

where k is the thermal conductivity of the material, which is a measure of the ability of a material to conduct heat.

$$q_x \propto A\frac{\Delta T}{\Delta x} \qquad q_x = -kA\frac{dT}{dx} \qquad \dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)} \qquad (1.2)$$

which is called **Fourier's law** of heat conduction. The dT/dx is the temperature gradient, which is the slope of the temperature curve on a T-x diagram at location x. The temperature gradient becomes negative when temperature decreases with increasing *x*. The negative sign in above equation ensures that heat transfer in the positive *x* direction is a positive quantity. Or for the heat flux;

$$q''_x = \frac{q_x}{A} = -k\frac{dT}{dx} \qquad (1.3)$$

A general statement of the conduction rate equation can be written as follows:

$$q_x = -k\,dy\,dz\frac{\partial T}{\partial x} \qquad q_y = -k\,dx\,dz\frac{\partial T}{\partial y} \qquad q_z = -k\,dx\,dy\frac{\partial T}{\partial z}$$

$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial x} + \boldsymbol{j}\frac{\partial T}{\partial y} + \boldsymbol{k}\frac{\partial T}{\partial z}\right) \qquad (1.4)$$

where ∇ is the three-dimensional del operator and *T(x, y, z)* is the scalar temperature field.

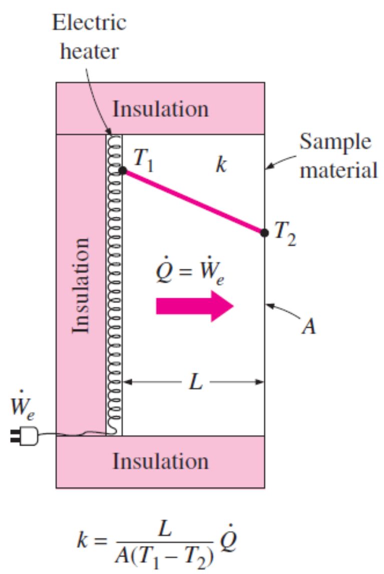
The **thermal conductivity** of a material is *the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference*. It is a measure of the material ability to conduct heat.

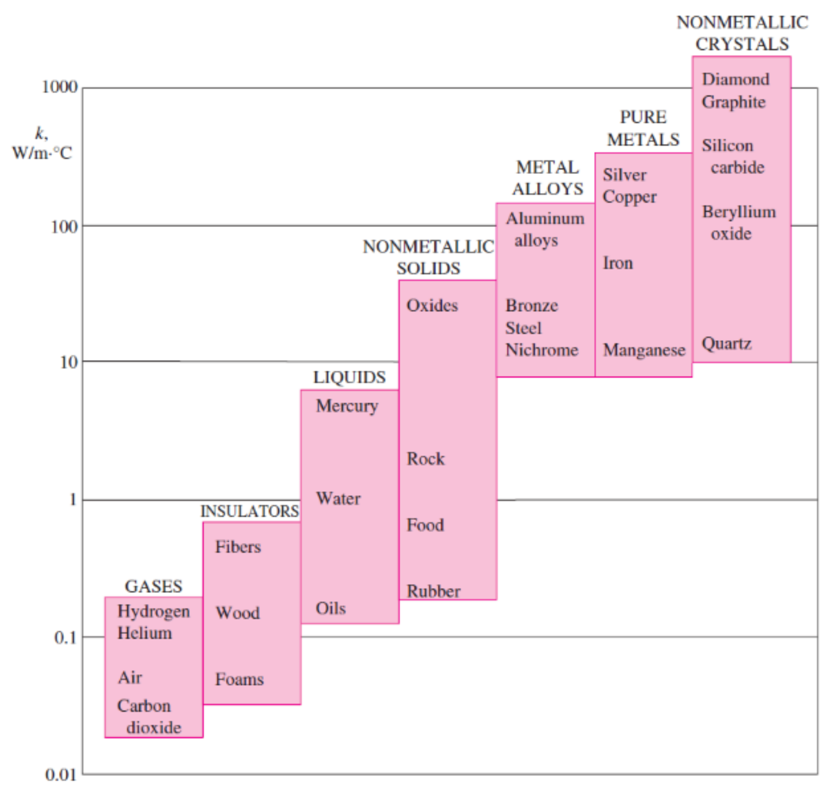
The thermal conductivity of some materials at room temperature

Material	k, W/m · °C*
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminum	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.613
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Glass fiber	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

The thermal conductivity can be measured experimentally as follows, see the Figure. A layer of material of known thickness and area can be heated from one side by an electric resistance heater. If the outer surfaces of the heater are well insulated, all the heat generated by the resistance heater will be transferred through the material whose conductivity is to be determined. Then measuring the two surface temperatures of the material when steady heat transfer is reached and substituting them into the equation shown together with other known quantities give the thermal conductivity.

The range of thermal conductivity of various materials at room temperature is shown in the following scheme.





The ratio of the thermal conductivity to the heat capacity is an important property termed the *thermal diffusivity* (α) which has units of m²/s:

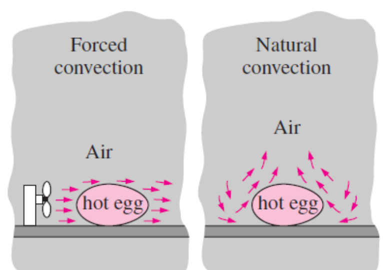
$$\alpha = \frac{k}{\rho c_p} \qquad (1.5)$$

It measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy. Materials of large α will respond quickly to changes in their thermal environment, whereas materials of small α will respond more sluggishly, taking longer time to reach a new equilibrium condition.

1.5.2 Convection Heat Transfer

Convection is *the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion*. The faster the fluid motion, the greater the convection heat transfer.

Convection is called *forced convection* if the fluid is forced to flow over the surface by external force such as a fan, pump, or the wind. In contrast, convection is called *natural* (or free) *convection* if the fluid motion is caused by buoyancy forces that are induced by density differences due to the variation of temperature in the fluid as shown in the Figure.



Despite the complexity of convection, the rate of convection heat transfer is observed to be proportional to the temperature difference, and is conveniently expressed by *Newton's law* of cooling as

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty) \quad \text{(W)} \qquad (1.6)$$

where *h* is the convection heat transfer coefficient in $W/m^2·°C$, A_s is the surface area through which convection heat transfer takes place, T_s is the surface temperature, and T_α is the temperature of the fluid sufficiently far from the surface. Note that at the surface, the fluid temperature equals the surface temperature of the solid.

The convection heat transfer coefficient *h* is not a property of the fluid as the thermal conductivity. Typical values of *h* are given in Table 1–5.

1.5.3 Radiation Heat Transfer

Radiation is the form of radiation emitted by bodies because of their temperature. Unlike conduction and convection, the transfer of energy by radiation does not require the presence of an intervening medium. All bodies at a temperature above absolute zero emit thermal radiation.

The maximum rate of radiation that can be emitted from a surface at an absolute temperature T_s in K is given by the *Stefan–Boltzmann law* as

$$\dot{Q}_{\text{emit, max}} = \sigma A_s T_s^4 \qquad \text{(W)} \tag{1.7}$$

Where $\sigma = 5.67\times10^{-8}\ W/m^2\cdot K^4$ is the Stefan–Boltzmann constant. The idealized surface that emits radiation at this maximum rate is called a blackbody, and the radiation emitted by a blackbody is called blackbody radiation. The radiation emitted by all real surfaces is less than the radiation emitted by a blackbody at the same temperature, and is expressed as

$$\dot{Q}_{emit} = \varepsilon\sigma A_s T_s^4 \qquad (W)$$

$$\dot{Q}_{rad} = \varepsilon\sigma A_s (T_s^4 - T_{surr}^4) \qquad (1.8)$$

where ε is the emissivity of the surface. The property emissivity, whose value is in the range $0 \leq \varepsilon \leq 1$, is a measure of how closely a surface approximates a blackbody for which $\varepsilon = 1$. The emissivities of some surfaces are given in Table 1–6.

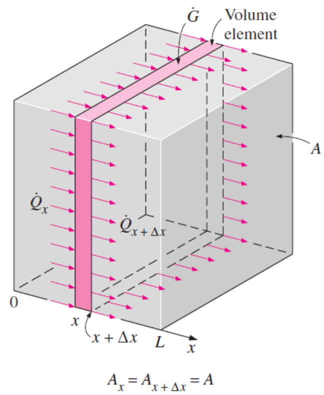
Chapter Two

Heat Conduction

2.1 Multi- and one-dimensional Heat Transfer

2.1.1 Heat Conduction Equation in a Large Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$



$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \tag{2.1}$$

The general Cartesian coordinate steady-state one-dimensional heat-conduction equation (no heat generation) is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \quad >> \quad \frac{d^2 T}{dx^2} = 0 \tag{2.2}$$

where the property ($\alpha = k/\rho c_p$) is the thermal diffusivity of the material and represents how fast heat propagates through a material.

The above equation is known as the **Fourier-Biot equation**, and it reduces to these forms under specified conditions:

(1) *Steady-state:* (called the **Poisson equation**)
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0$$

(2) *Transient, no heat generation:* (called the **diffusion equation**)
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.3)$$

(3) *Steady-state, no heat generation:* (called the **Laplace equation**)
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

(1) *Steady-state:* $(\partial/\partial t = 0)$
$$\frac{d^2 T}{dx^2} + \frac{\dot{g}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{g} = 0)$
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{g} = 0)$
$$\frac{d^2 T}{dx^2} = 0$$

General, one dimensional:

No generation (0), Steady-state (0)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Steady, one-dimensional:

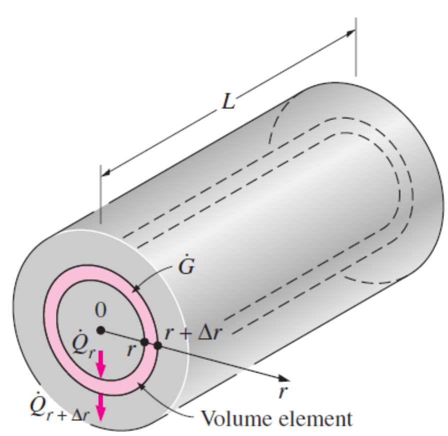
$$\frac{d^2 T}{dx^2} = 0$$

(2.4)

2.1.2 Heat Conduction Equation in a Long Cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.5)$$

It can be reduced to the following forms under specified conditions:



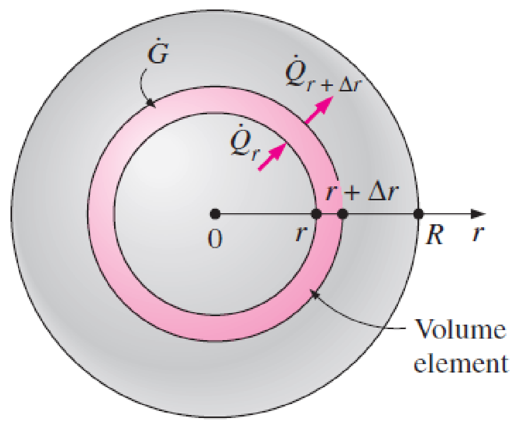
(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{g} = 0)$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.6)$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{g} = 0)$ $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

2.1.3 Heat Conduction Equation in a Sphere

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.7)$$



(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$

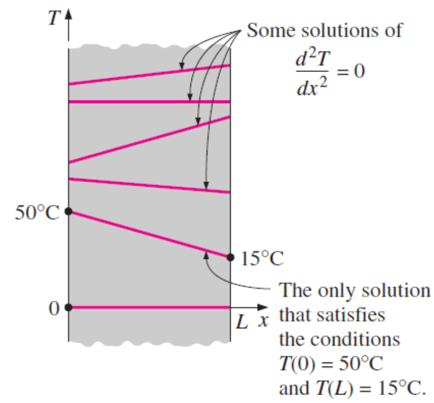
(2) *Transient, no heat generation:* $(\dot{g} = 0)$ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{g} = 0)$ $$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

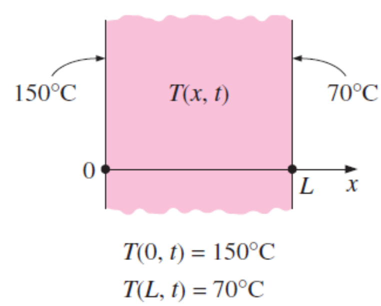
(2.8)

2.2 Boundary and Initial Conditions

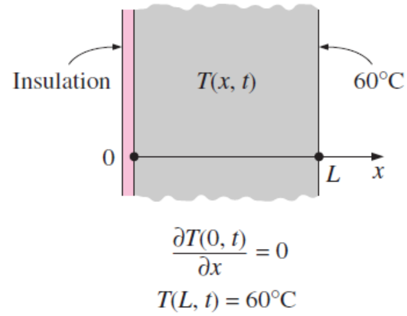
The mathematical expressions of the thermal conditions at the boundaries are called the ***boundary conditions***. To describe a heat transfer problem completely, *two boundary conditions* must be given for each direction of the coordinate system along which heat transfer is significant. Thus, four boundary conditions for two-dimensional problems, and six boundary conditions for three-dimensional problems are required. Boundary conditions most commonly used in practice are the specified temperature, specified heat flux, convection, and radiation boundary conditions.



A condition which is usually specified at time t = 0, is called the ***initial condition***, which is a mathematical expression for the temperature distribution of the medium *initially*. Note that we need only *one initial condition* for a heat conduction problem regardless of the dimension since the conduction equation is first order in time (it involves the first derivative of temperature with respect to time). Note that under ***steady conditions***, the heat conduction equation does not involve any time derivatives, and thus we do not need to specify an initial condition.



Some surfaces are commonly *insulated* in practice in order to minimize heat loss (or gain) with the surrounding. Insulation reduces heat transfer but does not totally eliminate it unless its thickness is *infinity*. However, heat transfer through a properly insulated surface can be taken to be zero since adequate insulation reduces heat transfer through a surface to negligible levels. Therefore, a well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at x = 0, for example) can be expressed as



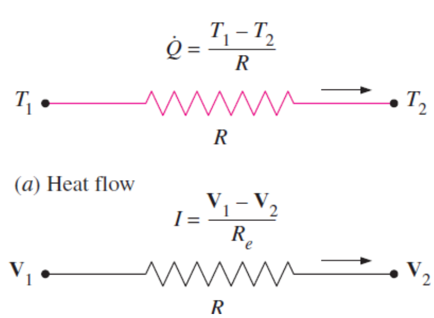
$$k \frac{\partial T(0, t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0, t)}{\partial x} = 0 \tag{2.9}$$

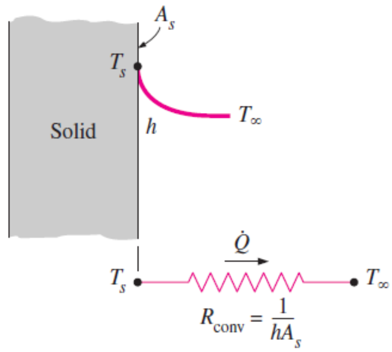
It is worthy to be mentioned that there are many other kinds of boundary conditions such as convection BC, radiation BC, interface BC, and etc.

2.3 Thermal resistance concept

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L} \quad >> \quad \dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \quad >> \quad R_{\text{wall}} = \frac{L}{kA} \tag{2.10}$$

It is the *thermal resistance* of the wall against heat conduction or simply the conduction resistance of the wall. Note that the thermal resistance of a medium depends on the geometry and the thermal properties of the medium.





$$\dot{Q}_{conv} = \frac{T_s - T_\infty}{R_{conv}} \quad \text{(W)}$$

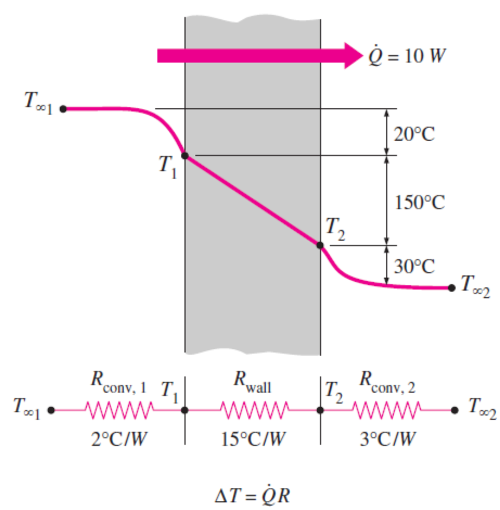
$$R_{conv} = \frac{1}{hA_s} \quad (°C/W)$$

Convection thermal resistance (2.11)

It is the thermal resistance of the surface against heat convection, or simply the convection resistance of the surface.

2.4 Thermal resistance network

$$\begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat conduction} \\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{from the wall}\end{pmatrix}$$



$$\dot{Q} = h_1 A(T_{\infty 1} - T_1) = kA\frac{T_1 - T_2}{L} = h_2 A(T_2 - T_{\infty 2})$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A} \quad (2.12)$$

$$= \frac{T_{\infty 1} - T_1}{R_{conv,1}} = \frac{T_1 - T_2}{R_{wall}} = \frac{T_2 - T_{\infty 2}}{R_{conv,2}}$$

$$\dot{Q} = \frac{T_{\infty} - T_{\infty 2}}{R_{\text{total}}} \qquad \text{(W)}$$

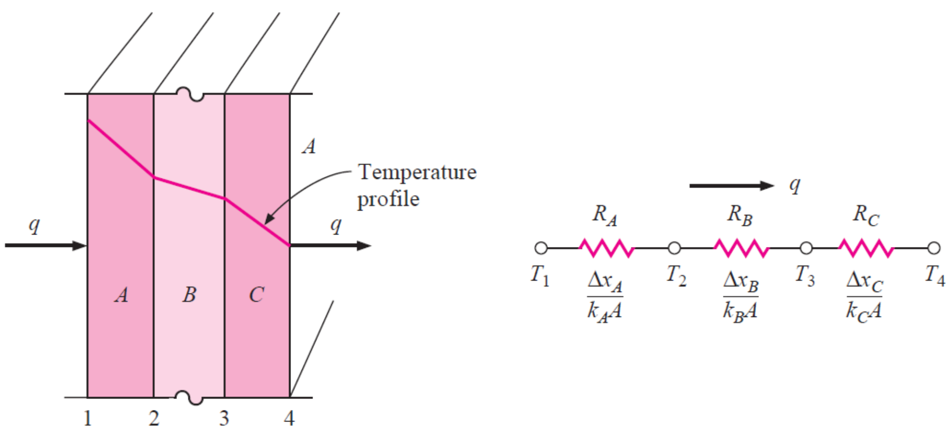
$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

2.5 Multi-layer wall

For the multilayer wall as shown in the Figure, the analysis would proceed as follows: The temperature gradients in the three materials are shown, and the heat flow may be written

$$q = -k_A A \frac{T_2 - T_1}{\Delta x_A} = -k_B A \frac{T_3 - T_2}{\Delta x_B} = -k_C A \frac{T_4 - T_3}{\Delta x_C} \tag{2.13}$$

Note that the heat flow must be the same through all sections.



Solving these three equations simultaneously, the heat flow is written

$$q = \frac{T_1 - T_4}{\Delta x_A / k_A A + \Delta x_B / k_B A + \Delta x_C / k_C A} \tag{2.14}$$

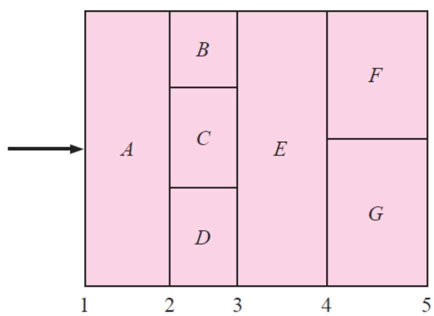
$$\text{Heat flow} = \frac{\text{thermal potential difference}}{\text{thermal resistance}} \qquad \text{or} \qquad q = \frac{\Delta T_{\text{overall}}}{\sum R_{\text{th}}}$$

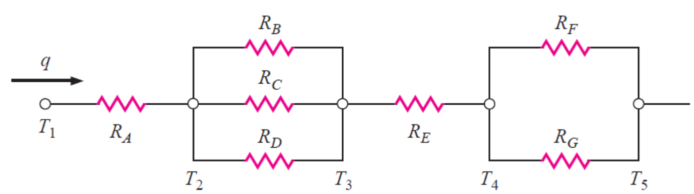
2.6 Insulation and R values

2.6.1 Plane walls

The thermal resistance, R value, can be written as

$$R = \frac{\Delta T}{q/A} \qquad {}^{\circ}\text{C} \cdot \text{m}^2/\text{W or } K.\text{m}^2/\text{W}$$





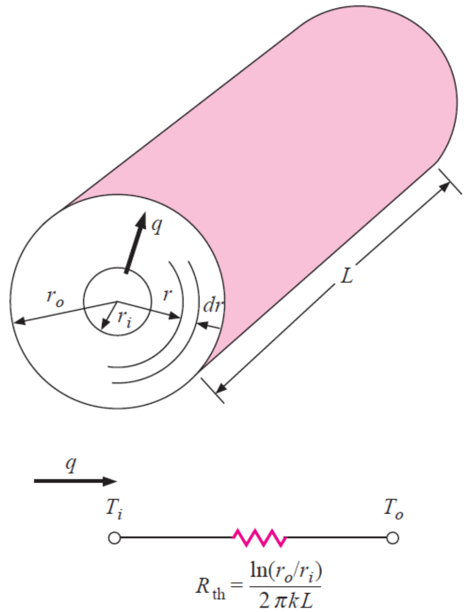
2.6.2 Cylindrical walls

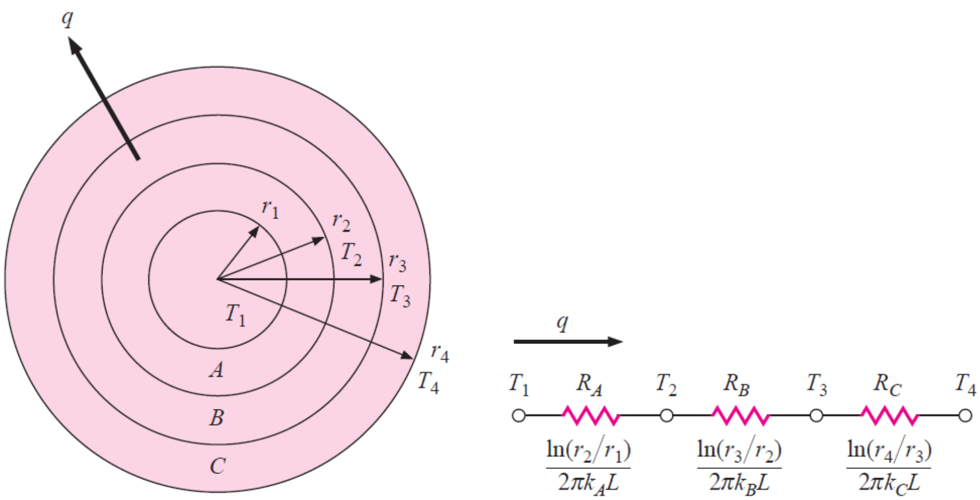
The circumferential area for heat flow in the cylindrical system is

$$A_r = 2\pi r L$$

so that Fourier's law is written

$$q_r = -kA_r \frac{dT}{dr} \qquad \text{or} \qquad q_r = -2\pi k r L \frac{dT}{dr} \tag{2.15}$$





with the boundary conditions;

$$T = T_i \qquad \text{at } r = r_i$$
$$T = T_o \qquad \text{at } r = r_o$$

$$q = \frac{2\pi kL\,(T_i - T_o)}{\ln(r_o/r_i)} \qquad >> \qquad R_{th} = \frac{\ln(r_o/r_i)}{2\pi kL} \qquad (2.16)$$

The thermal-resistance concept may be used for multiple-layer cylindrical walls just as it was used for plane walls. For the three-layer system shown in the above Figure the solution is

$$q = \frac{2\pi L\,(T_1 - T_4)}{\ln(r_2/r_1)/k_A + \ln(r_3/r_2)/k_B + \ln(r_4/r_3)/k_C} \tag{2.17}$$

2.6.3 Spherical walls

Spherical systems may also be treated as one-dimensional when the temperature is a function of radius only. The heat flow is then

$$q = \frac{4\pi k\,(T_i - T_o)}{1/r_i - 1/r_o} \tag{2.18}$$

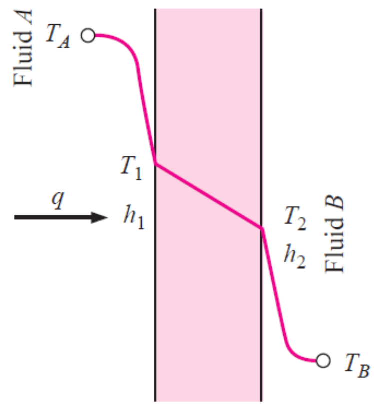
2.7 Summary of one-dimensional conduction

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

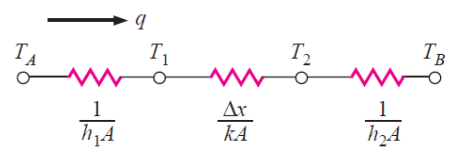
2.8 The overall heat-transfer coefficient

The overall heat transfer is calculated as the ratio of the overall temperature difference to the sum of the thermal resistances:



$$q = \frac{T_A - T_B}{1/h_1 A + \Delta x/kA + 1/h_2 A} \tag{2.19}$$

The overall heat transfer by combined conduction and convection is frequently expressed in terms of an overall heat-transfer coefficient *U*, defined by the relation:



$$q = UA\,\Delta T_{\text{overall}} \tag{2.20}$$

where A is some suitable area for the heat flow. The *overall heat-transfer coefficient* would be

$$U = \frac{1}{1/h_1 + \Delta x/k + 1/h_2} \tag{2.21}$$

The *overall heat-transfer coefficient* is also related to the R value through

$$U = \frac{1}{R\ \text{value}} \tag{2.22}$$

For a hollow cylinder exposed to a convection environment on its inner and outer surfaces, T_A and T_B are the two fluid temperatures.

Note that the area for convection is not the same for both fluids in this case, these areas depending on the inside tube diameter and wall thickness. The overall heat transfer would be expressed by

$$q = \frac{T_A - T_B}{\dfrac{1}{h_i A_i} + \dfrac{\ln(r_o/r_i)}{2\pi k L} + \dfrac{1}{h_o A_o}} \tag{2.23}$$

$$U_i = \frac{1}{\dfrac{1}{h_i} + \dfrac{A_i \ln(r_o/r_i)}{2\pi k L} + \dfrac{A_i}{A_o}\dfrac{1}{h_o}} \tag{2.24}$$

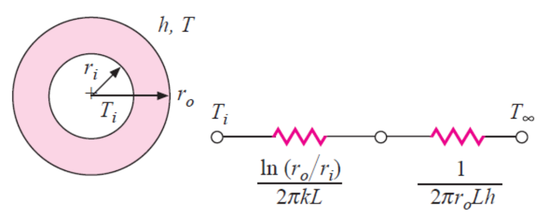
$$U_o = \frac{1}{\dfrac{A_o}{A_i}\dfrac{1}{h_i} + \dfrac{A_o \ln(r_o/r_i)}{2\pi k L} + \dfrac{1}{h_o}} \tag{2.25}$$

The general notion, for either the plane wall or cylindrical coordinate system, is that

$$UA = 1/\Sigma \mathrm{R_{th}} = 1/\mathrm{R_{th,overall}} \tag{2.26}$$

2.9 Critical thickness of insulation

Now let us manipulate this expression to determine the outer radius of insulation ro, which will maximize the heat transfer. The maximization condition is



$$q = \frac{2\pi L\,(T_i - T_\infty)}{\dfrac{\ln\,(r_o/r_i)}{k} + \dfrac{1}{r_o h}} \tag{2.27}$$

$$\frac{dq}{dr_o} = 0 = \frac{-2\pi L\,(T_i - T_\infty)\left(\dfrac{1}{kr_o} - \dfrac{1}{hr_o^2}\right)}{\left[\dfrac{\ln\,(r_o/r_i)}{k} + \dfrac{1}{r_o h}\right]^2} \tag{2.28}$$

yields;

$$r_o = \frac{k}{h} \tag{2.29}$$

The last equation expresses the critical-radius-of-insulation concept. If the outer radius is less than the value given by this equation, then the heat transfer will be increased by adding more insulation. For outer radii greater than the critical value an increase in insulation thickness will cause a decrease in heat transfer. The central concept is that for sufficiently small values of h the convection heat loss may actually increase with the addition of insulation because of increased surface area.

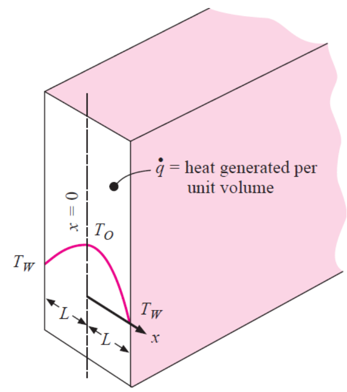
2.10 Heat-Source Systems

Many interesting systems deal with heat generated internally. Nuclear reactors are one example; electrical conductors and chemically reacting systems are others.

2.10.1 Plane Wall with Heat Sources (Heat Generation)

Many practical heat transfer applications involve the conversion of some form of energy into thermal energy in the medium. For example, resistance heating in wires, chemical reactions in a solid, and nuclear reactions in nuclear fuel rods where electrical, chemical, and nuclear energies are converted to heat.

Consider the plane wall with uniformly distributed heat sources shown in the figure. The thickness of the wall in the *x* direction is 2L, and it is assumed that the dimensions in the other directions are sufficiently large that the heat flow may be considered as one-dimensional. The heat generated per unit volume is $\dot{q}$, and we assume that the thermal conductivity does not vary with temperature. This situation might be produced in a practical situation by passing a current through an electrically conducting material.



The differential equation that governs the heat flow is:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{2.30}$$

By applying the B.C.,

$T = T_w, \; x = \pm L$

$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Because the temperature must be the same on each side of the wall, C1 must be zero. The temperature at the mid-plane (x = 0) is denoted by T_0 and from Equation $T_o = C_2$. The temperature distribution is therefore;

$$T - T_0 = -\frac{\dot{q}}{2k}x^2$$

If T = T_w at x = L (wall side);

$$T_0 = \frac{\dot{q}L^2}{2k} + T_w \qquad (2.31)$$

2.10.2 Cylinder with Heat Sources

$$T - T_w = \frac{\dot{q}}{4k}(R^2 - r^2) \qquad (2.32)$$

where T_0 is the temperature at r = 0 and is given by

$$T_0 = \frac{\dot{q}R^2}{4k} + T_w \qquad (2.33)$$

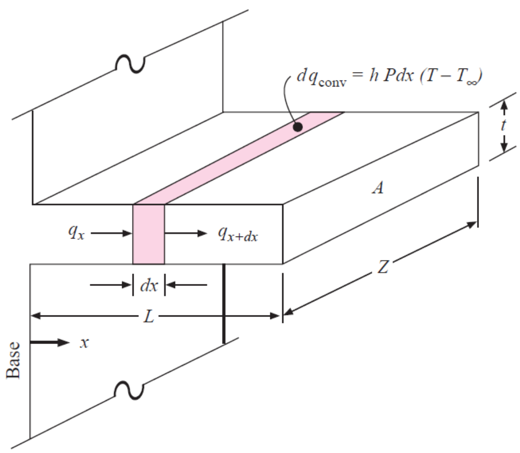
2.10.3 Sphere with heat source

$$T_o = \frac{\dot{q}R^2}{6k} + T_w \qquad (2.34)$$

2.11 Conduction-Convection Systems (Heat transfer from extended surfaces "Fins")

The heat that is conducted through a body must frequently be removed (or delivered) by some convection process. For example, the heat lost by conduction through a furnace wall must be dissipated to the surroundings through convection. In heat-exchanger applications a finned-tube arrangement might be used to remove heat from a hot liquid. The heat transfer from the liquid to the finned tube is by convection. The heat is conducted through the material and finally dissipated to the surroundings by convection. Obviously, an analysis of combined conduction-convection systems is very important from a practical standpoint.

[Energy in left face = energy out right face + energy lost by convection]



The defining equation for the convection heat-transfer coefficient is recalled as:

$$q = hA\,(T_w - T_\infty)$$

where A is the surface area for convection. Let the cross-sectional area of the fin be A and the perimeter be P. Then the energy quantities are

$$\text{Energy in left face} = q_x = -kA\frac{dT}{dx}$$

$$\text{Energy out right face} = q_{x+dx} = -kA\left.\frac{dT}{dx}\right]_{x+dx} \tag{2.35}$$

$$= -kA\left(\frac{dT}{dx} + \frac{d^2T}{dx^2}\,dx\right)$$

$$\text{Energy lost by convection} = hP\,dx\,(T - T_\infty)$$

When we combine the quantities, the energy balance yields

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$$

Let $\theta = T - T_\infty$.

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0$$

One boundary condition is

$$\theta = \theta_0 = T_0 - T_\infty \qquad \text{at } x = 0$$

The other boundary condition depends on the physical situation. Several cases may be considered:

CASE 1: The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

CASE 2: The fin is of finite length and loses heat by convection from its end.

CASE 3: The end of the fin is insulated so that $dT/dx = 0$ at $x = L$.

If we let $m^2 = hP/kA$, the general solution for the equation may be written as

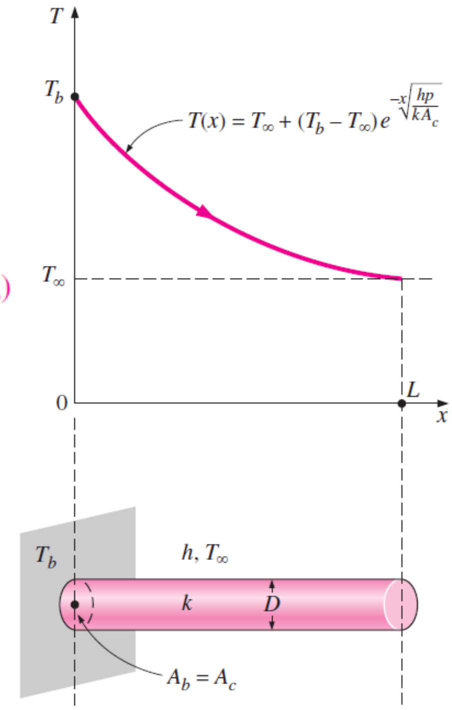
$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

For case 1 (Infinitely Long uniform Fin ($T_{fin\ tip} = T_\infty$)):

$$\begin{aligned} \theta &= \theta_0 \quad \text{at } x = 0 \\ \theta &= 0 \quad \text{at } x = \infty \end{aligned} \quad >> \quad \frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx}$$

very long fin $$\frac{T(x)-T_\infty}{T_b-T_\infty}=e^{-ax}=e^{-x\sqrt{hp/kA_c}}$$

$$\dot{Q}_{\text{long fin}}=-kA_c\left.\frac{dT}{dx}\right|_{x=0}=\sqrt{hpkA_c}\,(T_b-T_\infty) \qquad (2.36)$$



($p = \pi D$, $A_c = \pi D^2/4$ for a cylindrical fin)

For case 3 (Negligible Heat Loss from the Fin Tip (Insulated fin tip, $\dot{Q}_{\text{fin tip}}$=0):

$$\theta=\theta_0 \quad \text{at } x=0$$
$$\frac{d\theta}{dx}=0 \quad \text{at } x=L$$

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh a(L-x)}{\cosh aL}$$

$$q=-kA\theta_0 m\left(\frac{1}{1+e^{-2mL}}-\frac{1}{1+e^{+2mL}}\right)$$

$$\dot{Q}_{\text{insulated tip}}=-kA_c\left.\frac{dT}{dx}\right|_{x=0}=\sqrt{hpkA_c}\,(T_b-T_\infty)\tanh aL$$

$$=\sqrt{hPkA}\,\theta_0\tanh mL$$

(2.37)

For case 2 (finite length fin):

$$q = \sqrt{hPkA}\,(T_0 - T_\infty)\,\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \qquad (2.38)$$

Note that the heat transfer relations for the very long fin and the fin with negligible heat loss at the tip differ by the factor tanh (aL), which approaches 1 as L becomes very large.

It has been assumed that the substantial temperature gradients occur only in the *x* direction. This assumption will be satisfied if the fin is sufficiently thin. For most fins of practical interest the error introduced by this assumption is less than 1%. It is worthwhile to note that the convection coefficient is seldom uniform over the entire surface, as has been assumed above. If severe non-uniform behavior is encountered, numerical finite-difference techniques must be employed to solve the problem.

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

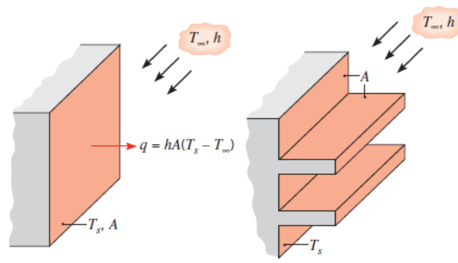
Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.75)	$M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.77)
B	Adiabatic: $d\theta/dx\|_{x=L} = 0$	$\frac{\cosh m(L-x)}{\cosh mL}$ (3.80)	$M \tanh mL$ (3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.82)	$M\frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.83)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx} (3.84)	M (3.85)

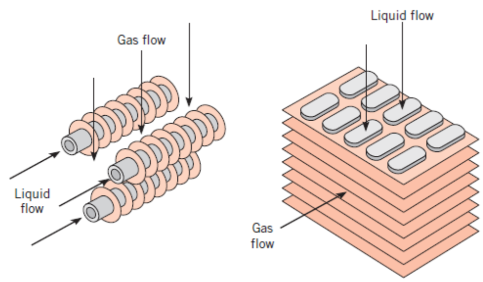
$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

2.11.1 Kinds of fins

The relations for the heat transfer were derived from a rod or fin of uniform cross-sectional area protruding from a flat wall. In practical applications, fins may have varying cross-sectional areas and may be attached to circular surfaces. In either case the area must be considered as a variable in the derivation, and solution of the basic differential equation and the mathematical techniques become more complex.





2.11.2 Fin efficiency

To indicate the effectiveness of a fin in transferring a given quantity of heat, a new parameter called *fin efficiency* is defined by

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\,max}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}} \quad (2.39)$$

Or

$$\dot{Q}_{fin} = \eta_{fin}\,\dot{Q}_{fin,\,max} = \eta_{fin}\,hA_{fin}\,(T_b - T_\infty) \quad (2.40)$$

For the cases of constant cross section of *very long fins* and *fins with insulated tips*, the fin efficiency can be expressed as, respectively;

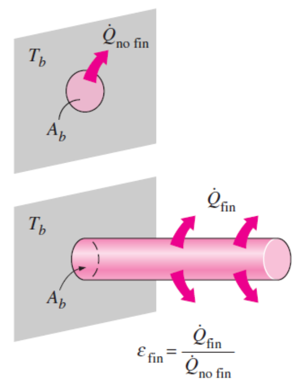
$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{aL} \tag{2.41}$$

$$\eta_{\text{insulated tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{\tanh aL}{aL} \tag{2.42}$$

since $A_{\text{fin}}=pL$ for fins with constant cross section.

2.11.3 Fin effectiveness

We should not expect to be able to maximize fin performance with respect to fin length. It is possible, however, to maximize the efficiency with respect to the quantity of fin material (mass, volume, or cost), and such a maximization process has rather obvious economic significance. The *fin effectiveness* can be defined as the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin, and it is estimated from



$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } base\ area\ A_b}{\text{Heat transfer rate from the surface of } area\ A_b} \tag{2.43}$$

An effectiveness of $\varepsilon_{\text{fin}} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted is equal to the heat convection. $\varepsilon_{\text{fin}} < 1$ indicates that the fin actually acts as insulation. This situation can occur when fins have low thermal conductivity. $\varepsilon_{\text{fin}} > 1$ indicates that fins are enhancing heat transfer from the surface, as they should. Finned surfaces are designed on the basis of maximizing effectiveness for a specified cost or minimizing cost for a desired effectiveness.

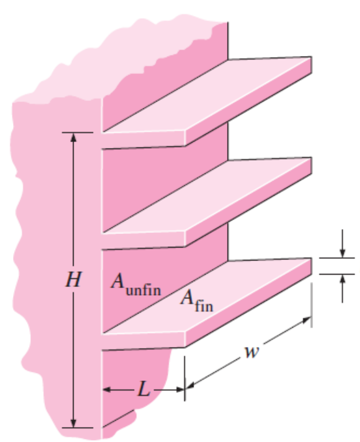
Note that both the fin efficiency and fin effectiveness are related to the performance of the fin, but they are different quantities. However, they are related to each other by

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}} \tag{2.44}$$

The effectiveness of a sufficiently long fin of uniform cross section under steady conditions is

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}} \tag{2.45}$$

When determining the rate of heat transfer from a finned surface, we must consider the unfinned portion of the surface as well as the fins. Therefore, the rate of heat transfer for a surface containing (n) fins can be expressed as



$A_{\text{no fin}} = w \times H$
$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$
$A_{\text{fin}} = 2 \times L \times w + t \times w$ (one fin)
$\approx 2 \times L \times w$

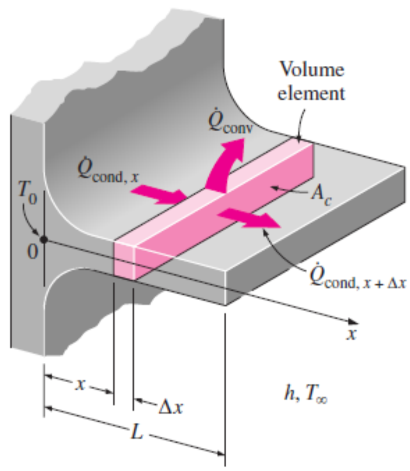
$$\begin{aligned}\dot{Q}_{\text{total, fin}} &= \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}} \\ &= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty) \\ &= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)\end{aligned} \tag{2.46}$$

The overall effectiveness is

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)} \quad (2.46)$$

2.11.4 Fins of non-uniform cross-sectional area

$$\dot{Q}_{\text{cond},\, x} = \dot{Q}_{\text{cond},\, x+\Delta x} + \dot{Q}_{\text{conv}}$$



where

$$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$

Substituting and dividing by Δx, we obtain

$$\frac{\dot{Q}_{\text{cond},\,x+\Delta x} - \dot{Q}_{\text{cond},\,x}}{\Delta x} + hp(T - T_\infty) = 0$$

Taking the limit as $\Delta x \rightarrow 0$ gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0$$

From Fourier's law of heat conduction we have

$$\dot{Q}_{\text{cond}} = -kA_c \frac{dT}{dx}$$

where A_c is the cross-sectional area of the fin at location x.

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0$$

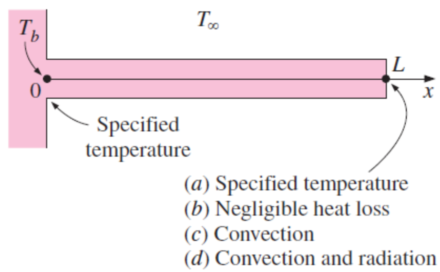
In the special case of constant cross section and constant thermal conductivity, it reduces to

$$\frac{d^2\theta}{dx^2} - a^2\theta = 0 \qquad \text{where} \qquad a^2 = \frac{hp}{kA_c} \qquad \theta = T - T_\infty$$

The solution functions of the differential equation above are the exponential functions e^{-ax} or e^{ax} or constant multiples of them. This can be verified by direct substitution. For example, the second derivative of e^{-ax} is a^2e^{-ax}, and its substitution into the last equation yields zero. Therefore, the general solution of the differential equation is

$$\theta(x) = C_1e^{ax} + C_2e^{-ax}$$

BC at fin base $\quad \theta(0) = \theta_b = T_b - T_\infty$

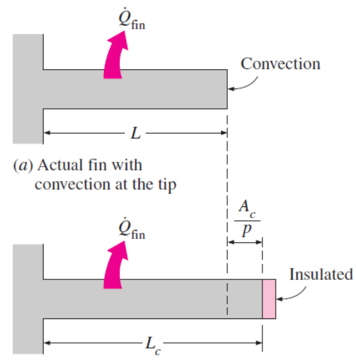


$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

This result provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used with conduction equation to calculate the conduction rate at any *x*.

2.11.5 Convection/(Convection and Radiation) from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that also includes the effects of radiation.



(b) Equivalent fin with insulated tip

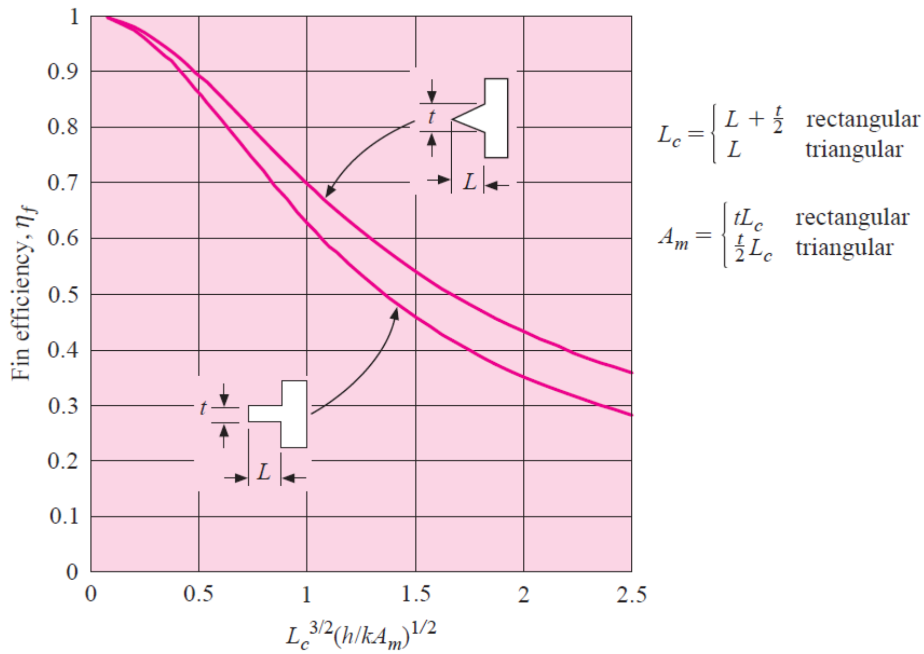
A practical way of accounting for the heat loss from the fin tip is to replace the fin length L in the relation for the insulated tip case by a corrected length defined as

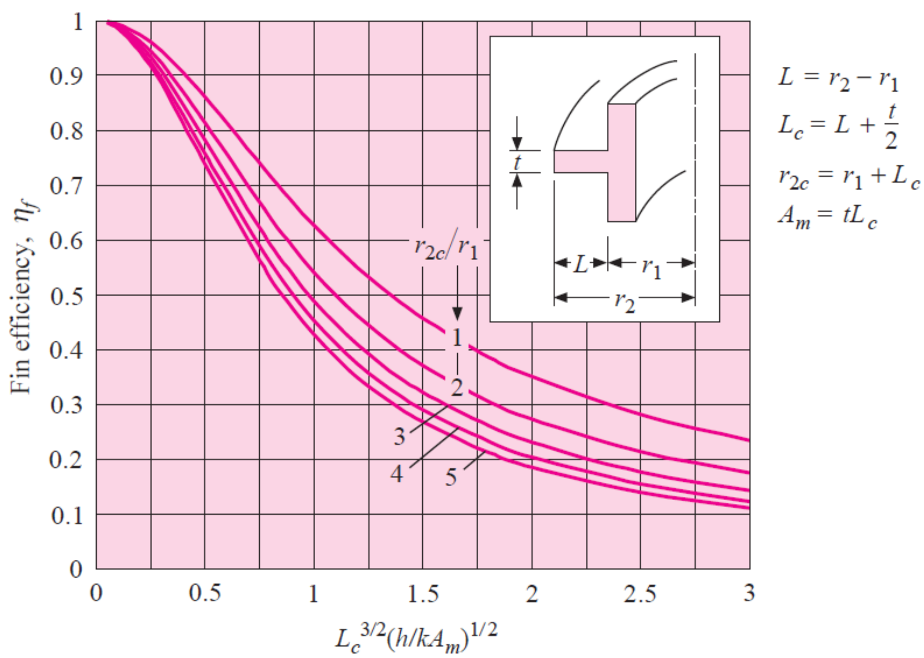
Corrected fin length: $$L_c = L + \frac{A_c}{p} \quad (2.47)$$

[Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip. Or, fins subjected to convection at their tips can be treated as fins with insulated tips by replacing the actual fin length by the corrected length].

Using the proper relations for A_c and p, the corrected lengths for rectangular and cylindrical fins are easily determined to be

$$L_{c,\text{ rectangular fin}} = L + \frac{t}{2} \quad \text{and} \quad L_{c,\text{ cylindrical fin}} = L + \frac{D}{4} \quad (2.48)$$





Efficiencies of straight rectangular, triangular fins, and circumferential fins of rectangular profile.

TABLE 3.5 Efficiency of common fin shapes

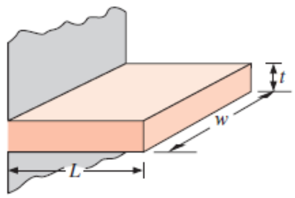
Straight Fins

Rectangular[a]

$A_f = 2wL_c$

$L_c = L + (t/2)$

$A_p = tL$

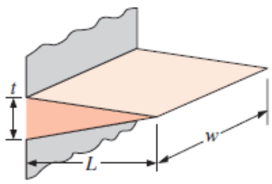


$$\eta_f = \frac{\tanh mL_c}{mL_c} \qquad (3.94)$$

Triangular[a]

$A_f = 2w[L^2 + (t/2)^2]^{1/2}$

$A_p = (t/2)L$



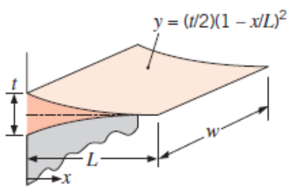
$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)} \qquad (3.98)$$

Parabolic[a]

$A_f = w[C_1 L +$
$(L^2/t)\ln(t/L + C_1)]$

$C_1 = [1 + (t/L)^2]^{1/2}$

$A_p = (t/3)L$



$$\eta_f = \frac{2}{[4(mL)^2 + 1]^{1/2} + 1} \qquad (3.99)$$

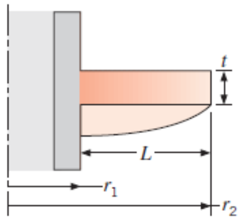
Circular Fin

Rectangular[a]

$A_f = 2\pi\,(r_{2c}^2 - r_1^2)$

$r_{2c} = r_2 + (t/2)$

$V = \pi\,(r_2^2 - r_1^2)t$



$$\eta_f = C_2 \frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})} \quad (3.96)$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

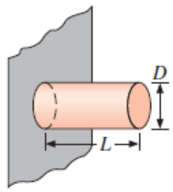
Pin Fins

Rectangular[b]

$A_f = \pi D L_c$

$L_c = L + (D/4)$

$V = (\pi D^2/4)L$

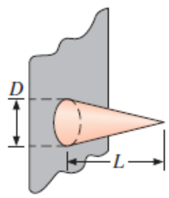


$$\eta_f = \frac{\tanh mL_c}{mL_c} \quad (3.100)$$

Triangular[b]

$A_f = \frac{\pi D}{2}[L^2 + (D/2)^2]^{1/2}$

$V = (\pi/12)D^2 L$



$$\eta_f = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)} \quad (3.101)$$

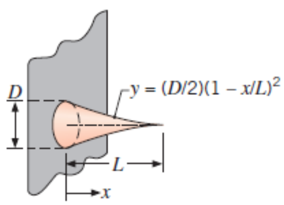
Parabolic[b]

$A_f = \frac{\pi L^3}{8D}\{C_3 C_4 - \frac{L}{2D}\ln[(2DC_4/L) + C_3]\}$

$C_3 = 1 + 2(D/L)^2$

$C_4 = [1 + (D/L)^2]^{1/2}$

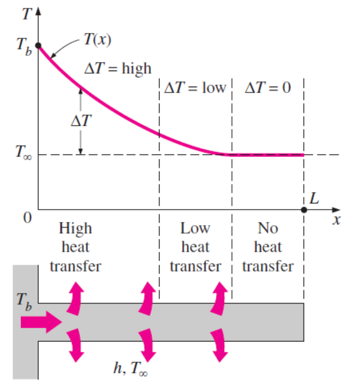
$V = (\pi/20)D^2 L$



$$\eta_f = \frac{2}{[4/9(mL)^2 + 1]^{1/2} + 1} \quad (3.102)$$

[a] $m = (2h/kt)^{1/2}$.
[b] $m = (4h/kD)^{1/2}$.

2.11.6 Proper Length of a Fin



An important step in the design of a fin is the determination of the appropriate length of the fin once the fin material and the fin cross section are specified. Longer fin, waste material, excessive weight, increased size, and thus increased cost with no benefit. To get a sense of the proper length of a fin, we compare heat transfer from a fin of finite length to heat transfer from an infinitely long fin under the same conditions. The ratio of these two heat transfers is

Heat transfer ratio:

$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh aL$$

The variation of heat transfer from a fin relative to that from an infinitely long fin

aL	$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh aL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

The heat transfer performance of heat sinks is usually expressed in terms of their thermal resistances R in °C/W, which is defined as

$$\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty)$$

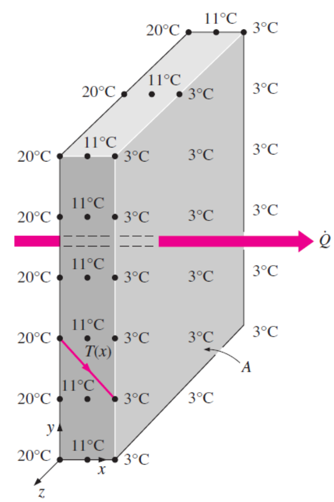
A small value of thermal resistance indicates a small temperature drop across the heat sink, and thus a high fin efficiency.

Chapter Three

Steady-State Conduction Multiple Dimensions

3.1 Introduction

It is preferable to analyze the more general case of two-dimensional heat flow. For steady state with no heat generation and assuming constant thermal conductivity, the Laplace equation applies



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (3.1)$$

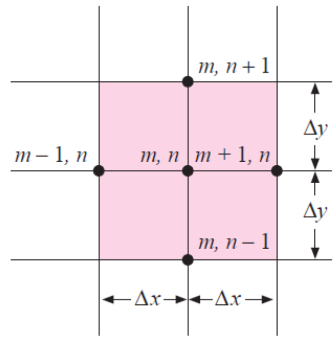
The objective of any heat-transfer analysis is usually to predict heat flow or the temperature that results from a certain heat flow. The solution to above equation gives the temperature in a two-dimensional coordinates; x and y. Then the heat flow in both directions may be calculated from the Fourier equations

$$q_x = -kA_x \frac{\partial T}{\partial x} \quad \text{and} \quad q_y = -kA_y \frac{\partial T}{\partial y} \qquad (3.2)$$

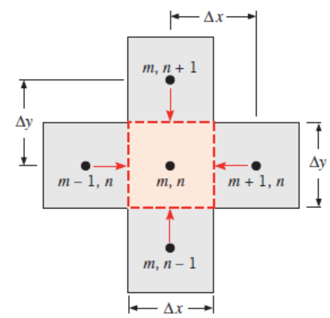
These heat-flow quantities are directed either in the x direction or in the y direction. The total heat flow at any point in the material is the resultant of the q_x and q_y at that point. So if the temperature distribution in the material is known, we may easily establish the heat flow.

3.2 Numerical method for analysis

An immense number of analytical solutions for conduction heat-transfer problems have been accumulated in the literature over the past 150 years. The most fruitful approach to the problem is one based on finite-difference methods (FDM).



Consider a two-dimensional body that is to be divided into equal increments in both the x and y directions as shown. The nodal points are designated the m locations indicating the x increment and the n locations indicating the y increment. We wish to establish the temperatures at any of these nodal points within the body using the above equation as a governing condition. FDM is used to approximate differential increments in the temperature and space coordinates; and the smaller we choose these finite increments, the more precise solution can be obtained.



The temperature gradients may be written as follows:

$$\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x} \qquad >> \qquad q_{(m+1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x} \qquad >> \qquad q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} \approx \frac{T_{m,n+1}-T_{m,n}}{\Delta y} \quad >> \quad q_{(m,n+1)\rightarrow(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n-1/2} \approx \frac{T_{m,n}-T_{m,n-1}}{\Delta y} \quad >> \quad q_{(m,n-1)\rightarrow(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}$$

(3.3)

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} \tag{3.4}$$

Thus the FDM becomes

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} = 0 \tag{3.5}$$

If $\Delta x = \Delta y$;

$$\boxed{T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0} \tag{3.6}$$

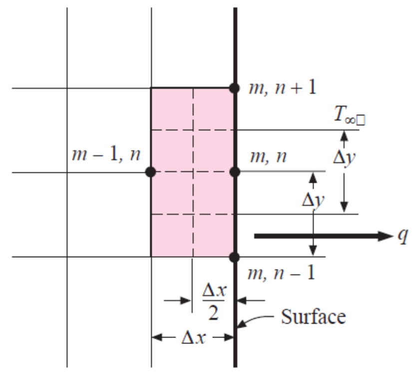
It is used for constant thermal conductivity. It states that the net heat flow into any node is zero at steady-state conditions. We can also devise a FDM to take heat generation into account. We merely add the term $\dot{q}/k$ into the general equation and obtain

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0 \tag{3.7}$$

If $\Delta x = \Delta y$;

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n-1} = 0 \tag{3.8}$$

When the solid is exposed to some convection boundary condition, the temperatures at the surface can be calculated as



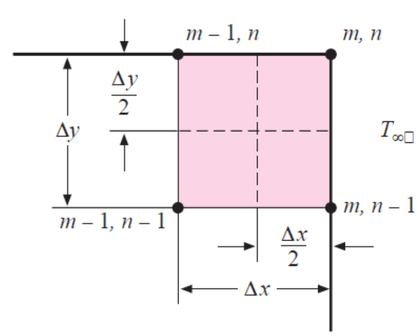
$$-k\,\Delta y\frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$
$$= h\,\Delta y(T_{m,n} - T_\infty)$$

If $\Delta x = \Delta y$;

$$T_{m,n}\left(\frac{h\,\Delta x}{k} + 2\right) - \frac{h\,\Delta x}{k}T_\infty - \frac{1}{2}(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) = 0 \tag{3.9}$$

The last equation must be written for each node along the surface in contact with the convection surface. While the interior nodes are solved using the first boundary condition (only conduction).

For the insulated wall or a corner exposed to a convection boundary condition he energy balance for the corner section is



$$-k\frac{\Delta y}{2}\frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h\frac{\Delta x}{2}(T_{m,n} - T_\infty) + h\frac{\Delta y}{2}(T_{m,n} - T_\infty)$$

If $\Delta x = \Delta y$;

$$2T_{m,n}\left(\frac{h\,\Delta x}{k}+1\right)-2\frac{h\,\Delta x}{k}T_{\infty}-(T_{m-1,n}+T_{m,n-1})=0 \qquad (3.10)$$

Other boundary conditions may be treated in a similar fashion as given in the following Table for different geometrical and boundary situations, while still using uniform increments in $\Delta x = \Delta y$.

Table 3-2 | Summary of nodal formulas for finite-difference calculations. (Dashed lines indicate element volume.)†

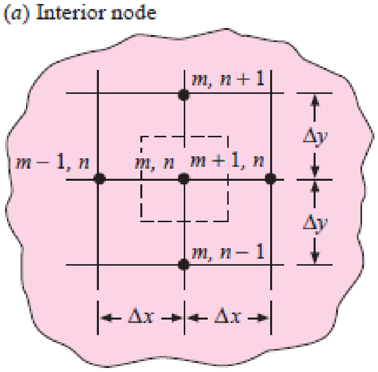
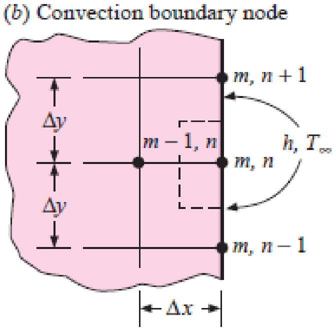
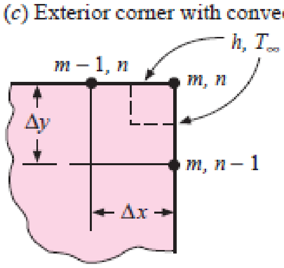
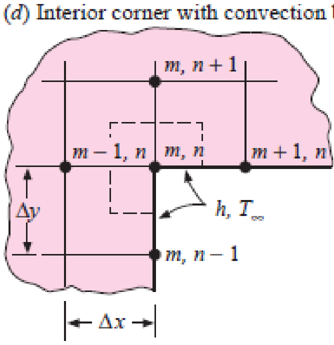
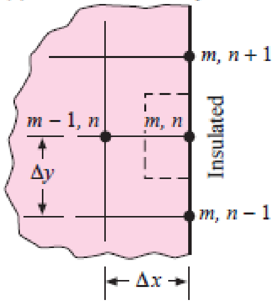
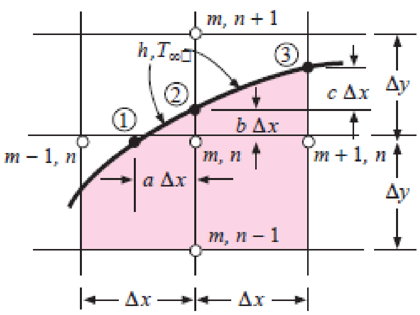
Physical situation	Nodal equation for equal increments in x and y (second equation in situation is in form for Gauss-Seidel iteration)
(*a*) Interior node 	$0 = T_{m+1,n} + T_{m,n+1} + T_{m-1,n} + T_{m,n-1} - 4T_{m,n}$ $T_{m,n} = (T_{m+1,n} + T_{m,n+1} + T_{m-1,n} + T_{m,n-1})/4$
(*b*) Convection boundary node 	$0 = \frac{h\Delta x}{k} T_\infty + \frac{1}{2}(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) - \left(\frac{h\Delta x}{k} + 2\right) T_{m,n}$ $T_{m,n} = \frac{T_{m-1,n} + (T_{m,n+1} + T_{m,n-1})/2 + \text{Bi}\ T_\infty}{2 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$
(*c*) Exterior corner with convection boundary 	$0 = 2\frac{h\Delta x}{k} T_\infty + (T_{m-1,n} + T_{m,n-1}) - 2\left(\frac{h\Delta x}{k} + 1\right) T_{m,n}$ $T_{m,n} = \frac{(T_{m-1,n} + T_{m,n-1})/2 + \text{Bi}\ T_\infty}{1 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$
(*d*) Interior corner with convection boundary 	$0 = 2\frac{h\Delta x}{k} T_\infty + 2T_{m-1,n} + T_{m,n+1} + T_{m+1,n} + T_{m,n-1} - 2\left(3 + \frac{h\Delta x}{k}\right) T_{m,n}$ $T_{m,n} = \frac{\text{Bi}\ T_\infty + T_{m,n+1} + T_{m-1,n} + (T_{m+1,n} + T_{m,n-1})/2}{3 + \text{Bi}}$ $\text{Bi} = \frac{h\Delta x}{k}$

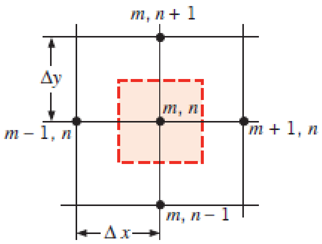
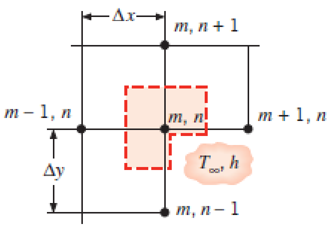
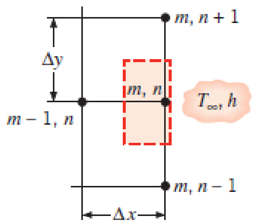
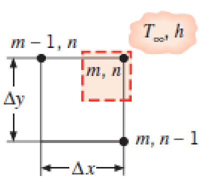
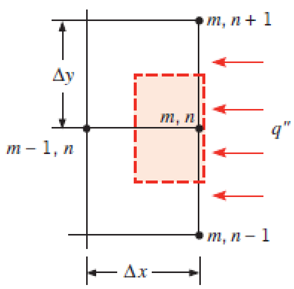
Table 3-2 | (*Continued*).

<table>
<tr><th>Physical situation</th><th>Nodal equation for equal increments in x and y
(second equation in situation is in form for Gauss-Seidel iteration)</th></tr>
<tr><td>(e) Insulated boundary

</td><td>$0 = T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n}$

$T_{m,n} = (T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n})/4$</td></tr>
<tr><td>(f) Interior node near curved boundary‡

</td><td>$0 = \frac{2}{b(b+1)}T_2 + \frac{2}{a+1}T_{m+1,n} + \frac{2}{b+1}T_{m,n-1} + \frac{2}{a(a+1)}T_1 - 2\left(\frac{1}{a} + \frac{1}{b}\right)T_{m,n}$</td></tr>
<tr><td>(g) Boundary node with convection along curved boundary—node 2 for (f) above§</td><td>$0 = \frac{b}{\sqrt{a^2+b^2}}T_1 + \frac{b}{\sqrt{c^2+1}}T_3 + \frac{a+1}{b}T_{m,n} + \frac{h\Delta x}{k}(\sqrt{c^2+1} + \sqrt{a^2+b^2})T_\infty$
$- \left[\frac{b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{c^2+1}} + \frac{a+1}{b} + (\sqrt{c^2+1} + \sqrt{a^2+b^2})\frac{h\Delta x}{k}\right]T_2$</td></tr>
</table>

†Convection boundary may be converted to insulated surface by setting $h = 0$ (Bi = 0).
‡This equation is obtained by multiplying the resistance by $4/(a+1)(b+1)$
§This relation is obtained by dividing the resistance formulation by 2.

TABLE 4.2 Summary of nodal finite-difference equations

Configuration	Finite-Difference Equation for $\Delta x = \Delta y$	
	$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$ **Case 1.** Interior node	(4.29)
	$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(3 + \frac{h\,\Delta x}{k}\right)T_{m,n} = 0$ **Case 2.** Node at an internal corner with convection	(4.41)
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 2\right)T_{m,n} = 0$ **Case 3.** Node at a plane surface with convection	(4.42)[a]
	$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 1\right)T_{m,n} = 0$ **Case 4.** Node at an external corner with convection	(4.43)
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\,\Delta x}{k} - 4T_{m,n} = 0$ **Case 5.** Node at a plane surface with uniform heat flux	(4.44)[b]

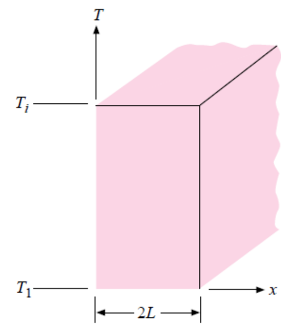
[a,b] To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

Chapter Four

Unsteady-state conduction (Transient)

4.1 Introduction

To analyze a transient conduction heat transfer, it can be done by solving the general heat-conduction equation by the separation-of-variables method, similar to the analytical treatment used for the two-dimensional steady-state problem. Consider the infinite plate of thickness 2L as shown. Initially the plate is at a uniform temperature T_i, and at time zero the surfaces are suddenly lowered to $T = T_1$. The differential equation is



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \quad \text{or} \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial \tau} \qquad \theta = T - T_1$$

with the initial and boundary conditions

$$\theta = \theta_i = T_i - T_1 \quad \text{at } \tau = 0,\ 0 \le x \le 2L$$

$$\theta = 0 \quad \text{at } x = 0,\ \tau > 0$$

$$\theta = 0 \quad \text{at } x = 2L,\ \tau > 0$$

The final series form of the solution is therefore

$$\theta = \sum_{n=1}^{\infty} C_n e^{-[n\pi/2L]^2 \alpha\tau} \sin\frac{n\pi x}{2L}$$

This equation may be recognized as a Fourier sine expansion with the constants C_n determined from the initial condition:

$$C_n = \frac{1}{L}\int_0^{2L} \theta_i \sin\frac{n\pi x}{2L}\,dx = \frac{4}{n\pi}\theta_i \qquad n = 1, 3, 5, \ldots$$

The final series solution is therefore

$$\frac{\theta}{\theta_i} = \frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-[n\pi/2L]^2\alpha\tau}\sin\frac{n\pi x}{2L} \qquad n = 1, 3, 5 \ldots$$

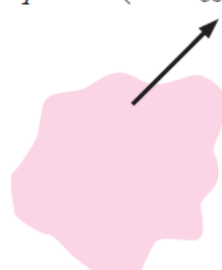
We note that at time zero (τ =0) the series on the right side of equation must converge to unity for all values of *x*.

4.2 Lumped heat capacity system

The transient heat conduction can be found by analyzing systems that might consider uniform in temperature. This type of analysis is called the *lumped-heat-capacity method.*

Lumped system analysis assumes a uniform temperature distribution throughout the body, which will be the case only when the thermal resistance of the body to heat conduction (the conduction resistance) is zero.

$$q = hA\,(T - T_\infty) = -c\rho V\,\frac{dT}{d\tau}$$



When a solid body is being heated by the hotter fluid surrounding it, heat is first convected to the body and subsequently conducted within the body. Such systems are idealized because a temperature gradient must exist in a material if heat is to be conducted into or out of the material. In general, the smaller the physical size of the body, the more realistic the assumption of a uniform temperature throughout; in the limit a differential volume could be employed as in the derivation of the general heat-conduction equation. The convection heat loss from the body is evidenced as a decrease in the internal energy of the body. Thus

$$q = hA(T - T_\infty) = -c\rho V \frac{dT}{d\tau}$$

where A is the surface area for convection and V is the volume. The initial condition is written

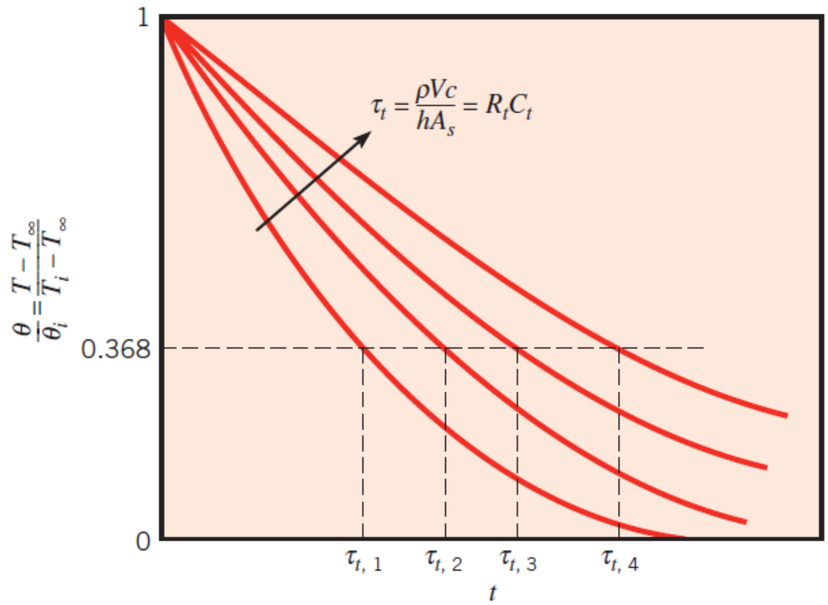
$$T = T_0 \qquad \text{at } \tau = 0$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho cV]\tau} \tag{4.1}$$

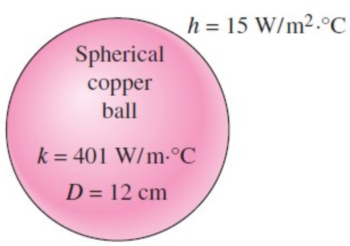
$$\frac{hA}{\rho cV} = \frac{1}{R_{th}C_{th}} \qquad R_{th} = \frac{1}{hA} \qquad C_{th} = \rho cV \tag{4.2}$$

The quantity ($\rho cV/hA$) is called the *time constant* of the system because it has the dimensions of time. When

$$\tau = \frac{c\rho V}{hA} \tag{4.3}$$



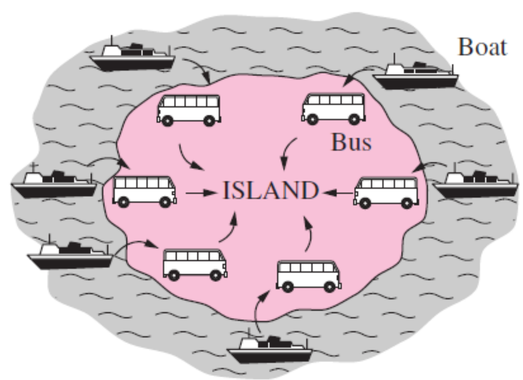
Thus, lumped system analysis is exact when Bi = 0 and approximate when Bi > 0. The smaller the Bi number, the more accurate the lumped system analysis. It is generally accepted that lumped system analysis is applicable if Bi ≤ 0.1



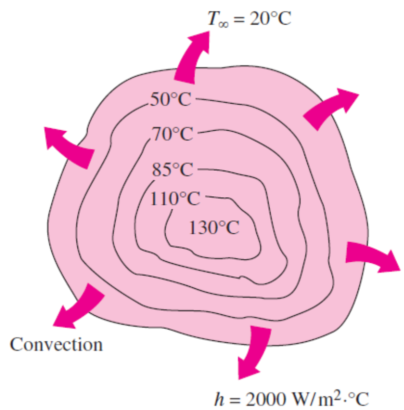
$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ \text{m}$$

$$\text{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

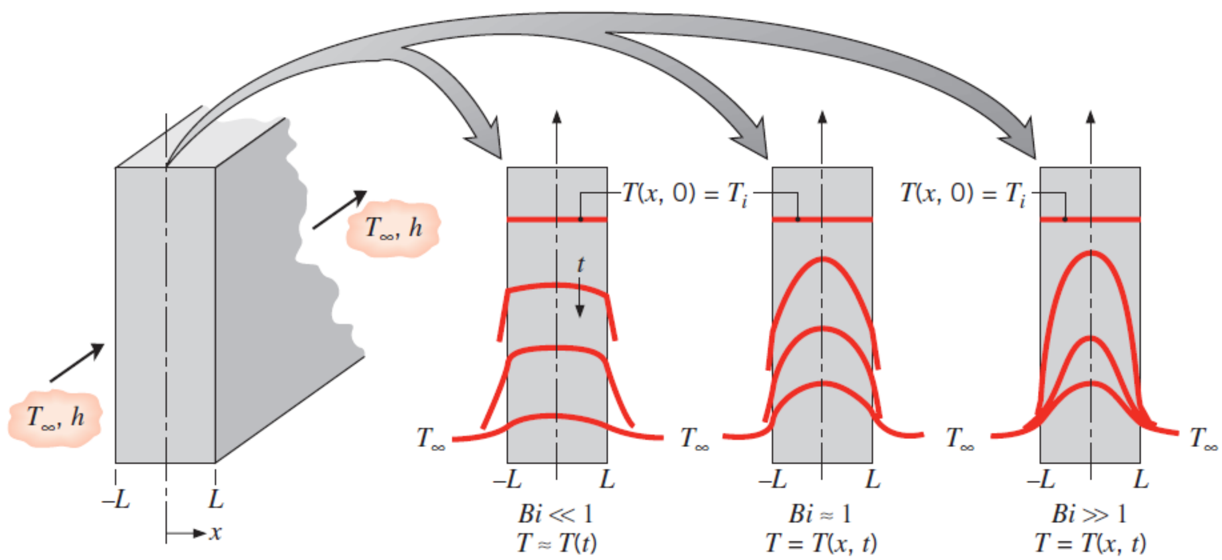
Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.



Example for illustrating the concept of Biot number.

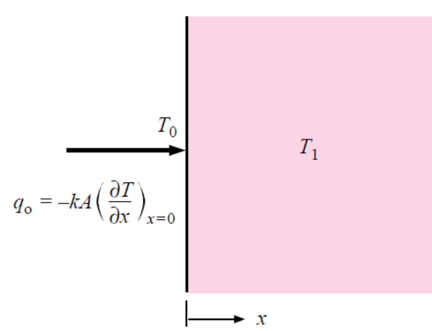


When the convection coefficient h is high and k is low, large temperature differences occur between the inner and outer regions of a large solid.



4.3 Transient heat flow in a semi-infinite solid

Consider the semi-infinite solid shown maintained at some initial temperature T_i. The surface temperature is suddenly lowered and maintained at a temperature T_0, and we seek an expression for the temperature distribution in the solid as a function of time. This temperature distribution may be used to calculate heat flow at any x position in the solid as a



function of time. For constant properties, the differential equation for the temperature distribution $T(x, \tau)$ is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

The boundary and initial conditions are

$$T(x, 0) = T_i$$
$$T(0, \tau) = T_0 \qquad \text{for } \tau > 0$$

This is a problem that may be solved by the Laplace-transform technique. At the surface ($x = 0$) the heat flow is

$$q_0 = \frac{kA(T_0 - T_i)}{\sqrt{\pi\alpha\tau}} \tag{4.4}$$

4.4 The Biot and Fourier Numbers

The dimensionless temperature profiles and heat flows may all be expressed in terms of two dimensionless parameters called the Biot and Fourier numbers:

$$\text{Biot number} = \text{Bi} = \frac{hs}{k} \tag{4.5}$$

$$\text{Fourier number} = \text{Fo} = \frac{\alpha\tau}{s^2} = \frac{k\tau}{\rho c s^2} \tag{4.6}$$

In these parameters s designates a characteristic dimension of the body; for the plate it is the half-thickness, whereas for the cylinder and sphere it is the radius. The Biot number compares the relative magnitudes of surface-convection and internal-conduction resistances to heat transfer. The Fourier modulus compares a characteristic body dimension with an approximate temperature-wave penetration depth for a given time τ.

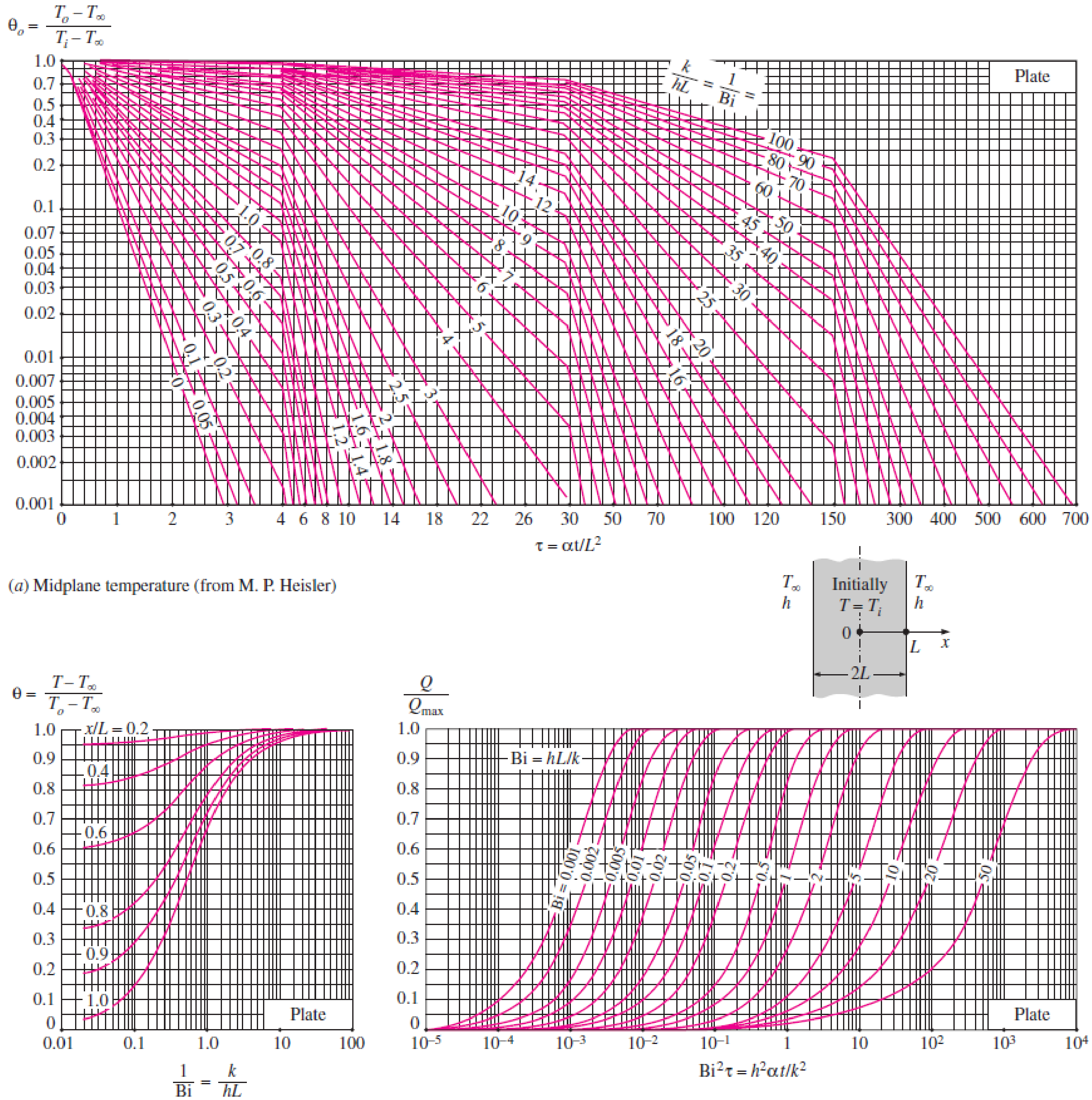
A very low value of the Biot modulus means that internal-conduction resistance is negligible in comparison with surface-convection resistance. This in turn implies that the temperature will be nearly uniform throughout the solid, and its behavior may be approximated by the lumped-capacity method of analysis. It is interesting to note that the exponent of the equation

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau} \quad >> \quad \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \tag{4.7}$$

thus $$b = \frac{hA_s}{\rho V C_p}$$

may be expressed in terms of the Biot and Fourier numbers if one takes the ratio V/A as the characteristic dimension s. Then,

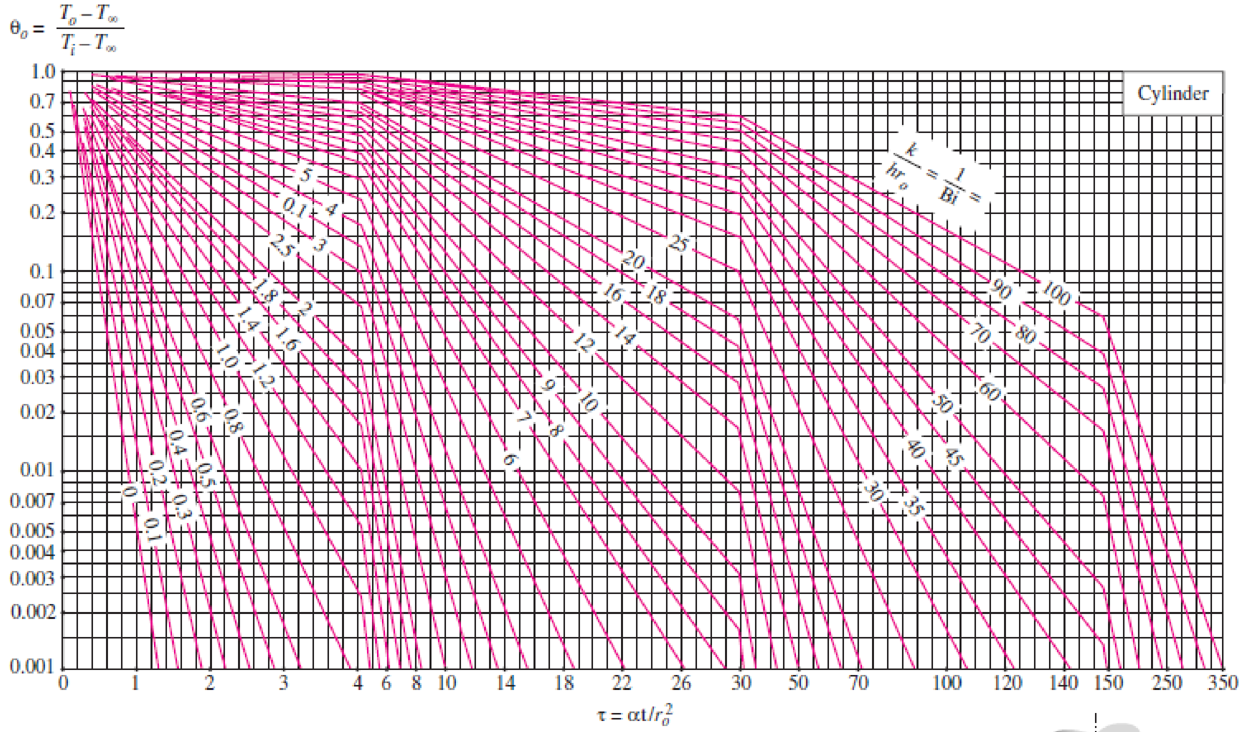
$$\frac{hA}{\rho c V}\tau = \frac{h\tau}{\rho c s} = \frac{hs}{k}\frac{k\tau}{\rho c s^2} = \text{Bi Fo} \tag{4.8}$$



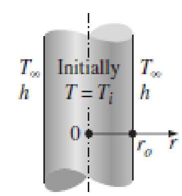
(b) Temperature distribution (from M. P. Heisler)

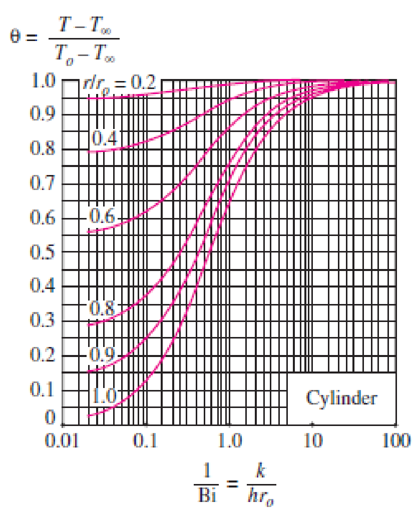
(c) Heat transfer (from H. Gröber et al.)

Figure 4-13

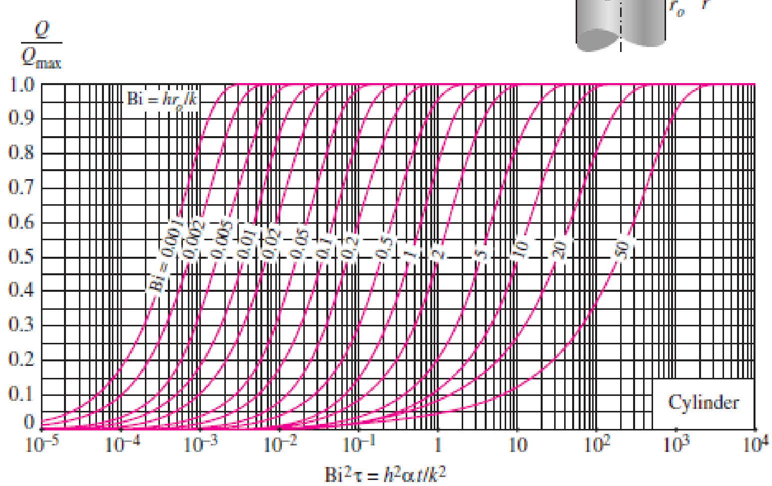


(*a*) Centerline temperature (from M. P. Heisler)





(*b*) Temperature distribution (from M. P. Heisler)



(*c*) Heat transfer (from H. Gröber et al.)

Figure 4-14

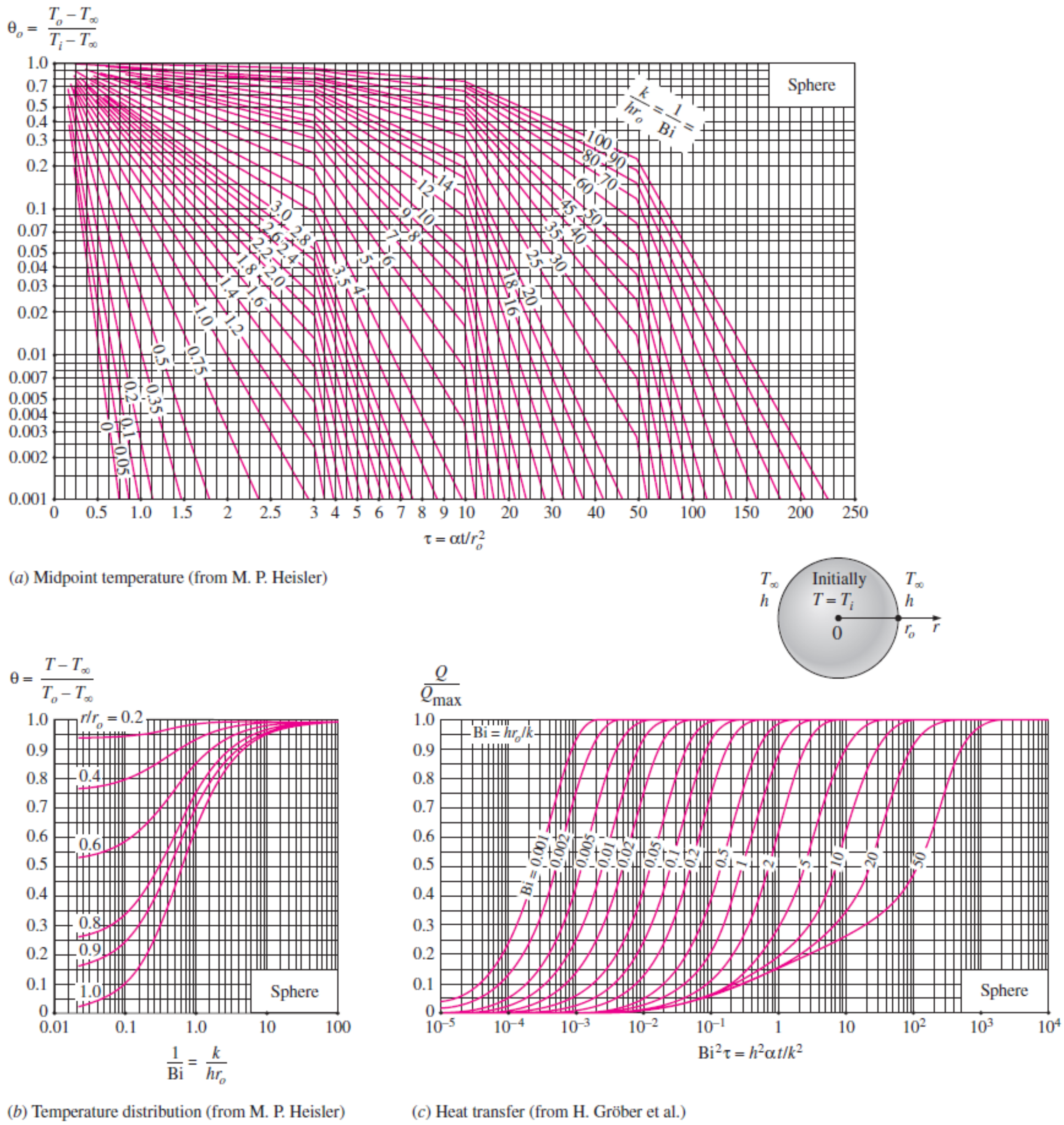


Figure 4-15

The fraction of heat transfer (actual heat Transfer) can also be determined from these relations, which are based on the one-term approximations already discussed:

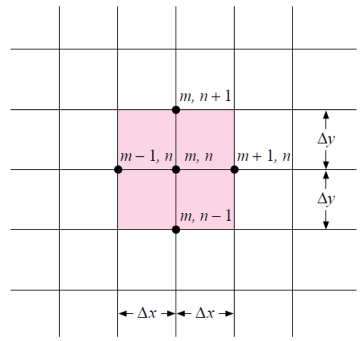
$$\text{Plane wall:} \quad \left(\frac{Q}{Q_{\max}}\right)_{\text{wall}} = 1 - \theta_{0,\text{ wall}} \frac{\sin \lambda_1}{\lambda_1}$$

$$\text{Cylinder:} \quad \left(\frac{Q}{Q_{\max}}\right)_{\text{cyl}} = 1 - 2\theta_{0,\text{ cyl}} \frac{J_1(\lambda_1)}{\lambda_1} \qquad (4.9)$$

$$\text{Sphere:} \quad \left(\frac{Q}{Q_{\max}}\right)_{\text{sph}} = 1 - 3\theta_{0,\text{ sph}} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$

4.5 Transient numerical method

Consider a two-dimensional body divided into increments as shown. The subscript m denotes the x position, and the subscript n denotes the y position. Within the solid body the differential equation that governs the heat flow is



$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \rho c \frac{\partial T}{\partial \tau} \qquad (4.10)$$

assuming constant properties. The second partial derivatives may be approximated by

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{1}{(\Delta x)^2}(T_{m+1,n} + T_{m-1,n} - 2T_{m,n})$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{1}{(\Delta y)^2}(T_{m,n+1} + T_{m,n-1} - 2T_{m,n}) \qquad (4.11)$$

The time derivative in the above first equation is approximated by

$$\frac{\partial T}{\partial \tau} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \qquad (4.12)$$

In this relation the superscripts designate the time increment. Combining the relations above gives the difference equation equivalent to above first equation

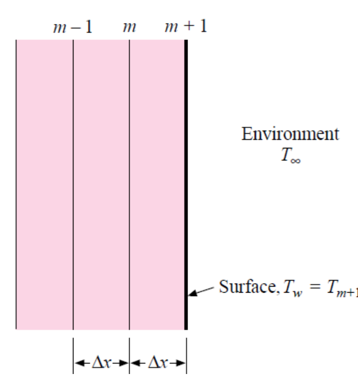
$$\frac{T^p_{m+1,n}+T^p_{m-1,n}-2T^p_{m,n}}{(\Delta x)^2}+\frac{T^p_{m,n+1}+T^p_{m,n-1}-2T^p_{m,n}}{(\Delta y)^2}=\frac{1}{\alpha}\frac{T^{p+1}_{m,n}-T^p_{m,n}}{\Delta\tau} \quad (4.13)$$

Thus, if the temperatures of the various nodes are known at any particular time, the temperatures after a time increment τ may be calculated by writing an equation like the last equation for each node and obtaining the values of $T^{p+1}_{m.n}$. The procedure may be repeated to obtain the distribution after any desired number of time increments. If $\Delta x = \Delta y$

$$T^{p+1}_{m,n}=\frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T^p_{m+1,n}+T^p_{m-1,n}+T^p_{m,n+1}+T^p_{m,n-1}\right)+\left[1-\frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right]T^p_{m,n} \quad (4.14)$$

The difference equations given above are useful for determining the internal temperature in a solid as a function of space and time.

At the boundary of the solid, a convection resistance to heat flow is usually involved, so that the above relations no longer apply. In general, each convection boundary condition must be handled separately, depending on the particular geometric shape under consideration. For case of the flat wall and for the one-dimensional system shown an energy balance will be made at the convection boundary:



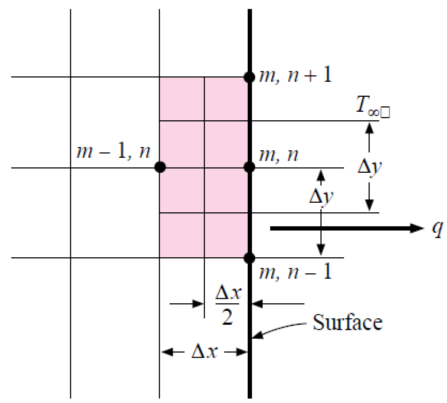
$$-kA\left.\frac{\partial T}{\partial x}\right]_{\text{wall}} = hA(T_w - T_\infty)$$

The finite-difference approximation would be given by

$$-k\frac{\Delta y}{\Delta x}(T_{m+1} - T_m) = h\,\Delta y(T_{m+1} - T_\infty)$$

$$T_{m+1} = \frac{T_m + (h\,\Delta x/k)T_\infty}{1 + h\,\Delta x/k} \tag{4.15}$$

We make a transient energy balance on the node (m, n) by setting the sum of the energy conducted and convected into the node equal to the increase in the internal energy of the node. Thus



$$k\,\Delta y\,\frac{T^p_{m-1,n} - T^p_{m,n}}{\Delta x} + k\,\frac{\Delta x}{2}\,\frac{T^p_{m,n+1} - T^p_{m,n}}{\Delta y} + k\,\frac{\Delta x}{2}\,\frac{T^p_{m,n-1} - T^p_{m,n}}{\Delta y}$$
$$+h\,\Delta y(T_\infty - T^p_{m,n}) = \rho c\,\frac{\Delta x}{2}\,\Delta y\,\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta\tau} \tag{4.16}$$

If $\Delta x = \Delta y$, the relation for $T^{p+1}_{m,n}$ becomes

$$T^{p+1}_{m,n} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left\{2\frac{h\,\Delta x}{k}\,T_\infty + 2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} + \left[\frac{(\Delta x)^2}{\alpha\,\Delta\tau} - 2\frac{h\,\Delta x}{k} - 4\right]T^p_{m,n}\right\} \tag{4.17}$$

The corresponding one-dimensional relation is

$$T_m^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left\{2\frac{h\,\Delta x}{k}T_\infty + 2T_{m-1}^p + \left[\frac{(\Delta x)^2}{\alpha\,\Delta\tau} - 2\frac{h\,\Delta x}{k} - 2\right]T_m^p\right\} \tag{4.18}$$

Notice now that the selection of the parameter $(\Delta x)^2/\alpha\Delta\tau$ is not as simple as it is for the interior nodal points because the heat-transfer coefficient influences the choice. It is still possible to choose the value of this parameter so that the coefficient of T_m^{p} or $T_{m,n}^{\mathrm{p}}$ will be zero.

4.6 Forward and Backward Differences

The equations above have been developed on the basis of a *forward-difference* technique in that the temperature of a node at a future time increment is expressed in terms of the surrounding nodal temperatures at the beginning of the time increment. The expressions are called *explicit formulations* because it is possible to write the nodal temperatures $T_{m,n}^{\mathrm{p+1}}$ explicitly in terms of the previous nodal temperatures $T_{m,n}^{\mathrm{p}}$. In this formulation, the calculation proceeds directly from one time increment to the next until the temperature distribution is calculated at the desired final state.

The difference equation may also be formulated by computing the space derivatives in terms of the temperatures at the p+1 time increment. Such an arrangement is called a *backward-difference* formulation because the time derivative moves backward from the times for heat conduction into the node. The equation is

$$\frac{T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}-2T_{m,n}^{p+1}}{(\Delta x)^2}+\frac{T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}-2T_{m,n}^{p+1}}{(\Delta y)^2}=\frac{1}{\alpha}\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta\tau} \tag{4.19}$$

$$T_{m,n}^{p}=\frac{-\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}+T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)+\left[1+\frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m,n}^{p+1} \tag{4.20}$$

We may now note that this backward-difference formulation does not permit the explicit calculation of the T^{p+1} in terms of T^{p}. Rather, a whole set of equations must be written for the entire nodal system and solved simultaneously to determine the temperatures T^{p+1}. Thus we say that the *backward-difference* method produces an *implicit* formulation for the future temperatures in the transient analysis.

The Biot and Fourier numbers may also be defined in the following way for problems in the numerical format

$$\mathrm{Bi}=\frac{h\,\Delta x}{k} \qquad \text{and} \qquad \mathrm{Fo}=\frac{\alpha\,\Delta\tau}{(\Delta x)^2}$$

By using this notation, the following tables have been constructed to summarize some typical nodal equations in both the explicit and implicit formulations. For the cases of $\Delta x = \Delta y$.

Table 4-2 | Explicit nodal equations. (Dashed lines indicate element volume.)†

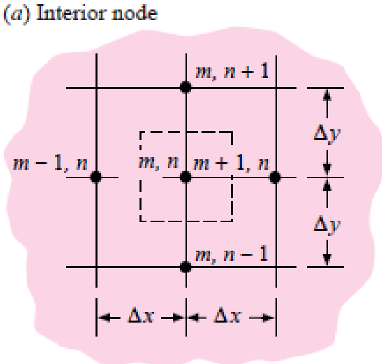
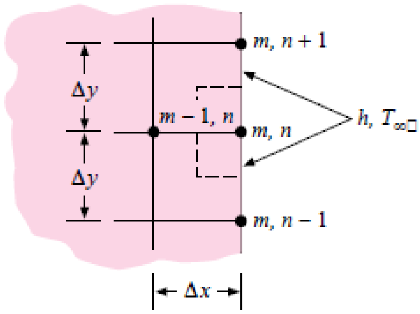
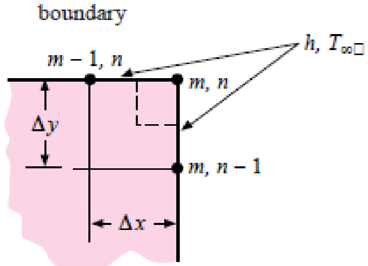
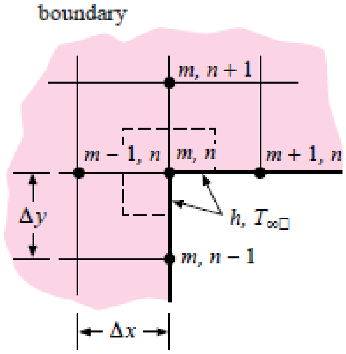
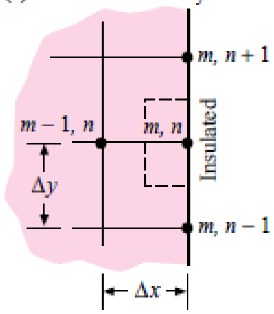
Physical situation	Nodal equation for $\Delta x = \Delta y$	Stability requirement
(*a*) Interior node 	$T^{p+1}_{m,n} = \text{Fo}\left(T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m+1,n} + T^p_{m,n-1}\right) + [1 - 4(\text{Fo})]T^p_{m,n}$ $T^{p+1}_{m,n} = \text{Fo}\left(T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m+1,n} + T^p_{m,n-1} - 4T^p_{m,n}\right) + T^p_{m,n}$	$\text{Fo} \le \frac{1}{4}$
(*b*) Convection boundary node 	$T^{p+1}_{m,n} = \text{Fo}\,[2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} + 2(\text{Bi})T^p_\infty] + [1 - 4(\text{Fo}) - 2(\text{Fo})(\text{Bi})]T^p_{m,n}$ $T^{p+1}_{m,n} = \text{Fo}\,[2\text{Bi}\,(T^p_\infty - T^p_{m,n}) + 2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} - 4T^p_{m,n}] + T^p_{m,n}$	$\text{Fo}(2 + \text{Bi}) \le \frac{1}{2}$
(*c*) Exterior corner with convection boundary 	$T^{p+1}_{m,n} = 2(\text{Fo})\,[T^p_{m-1,n} + T^p_{m,n-1} + 2(\text{Bi})T^p_\infty] + [1 - 4(\text{Fo}) - 4(\text{Fo})(\text{Bi})]T^p_{m,n}$ $T^{p+1}_{m,n} = 2\text{Fo}\,[T^p_{m-1,n} + T^p_{m,n-1} - 2T^p_{m,n} + 2\text{Bi}(T^p_\infty - T^p_{m,n})] + T^p_{m,n}$	$\text{Fo}(1 + \text{Bi}) \le \frac{1}{4}$
(*d*) Interior corner with convection boundary 	$T^{p+1}_{m,n} = \frac{2}{3}(\text{Fo})\,[2T^p_{m,n+1} + 2T^p_{m+1,n} + 2T^p_{m-1,n} + T^p_{m,n-1} + 2(\text{Bi})T^p_\infty] + [1 - 4(\text{Fo}) - \frac{4}{3}(\text{Fo})(\text{Bi})]T^p_{m,n}$ $T^{p+1}_{m,n} = (4/3)\text{Fo}\,[T^p_{m,n+1} + T^p_{m+1,n} + T^p_{m-1,n} - 3T^p_{m,n} + \text{Bi}\,(T^p_\infty - T^p_{m,n})] + T^p_{m,n}$	$\text{Fo}(3 + \text{Bi}) \le \frac{3}{4}$

Table 4-2 | (*Continued*).

Physical situation	Nodal equation for $\Delta x = \Delta y$	Stability requirement
(*e*) Insulated boundary 	$T_{m,n}^{p+1} = \text{Fo}\,[2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}] + [1-4(\text{Fo})]T_{m,n}^{p}$	$\text{Fo} \le \frac{1}{4}$

†Convection surfaces may be made insulated by setting $h = 0$ (Bi = 0).

Table 4-3 | Implicit nodal equations. (Dashed lines indicate volume element.)

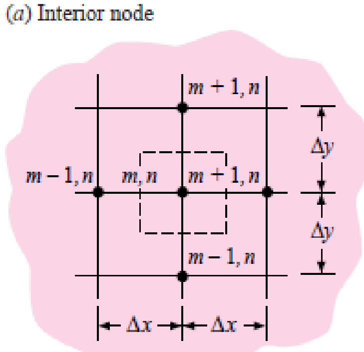
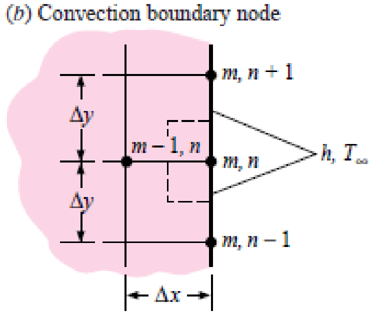
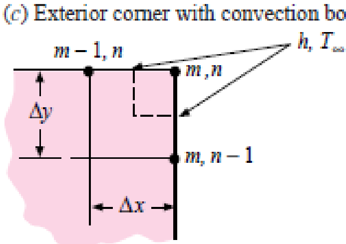
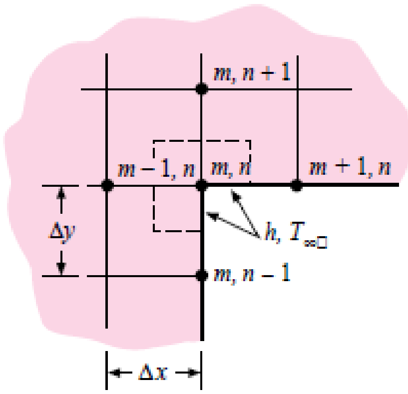
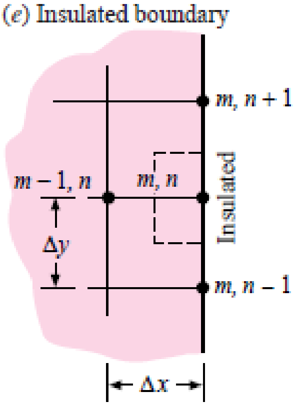
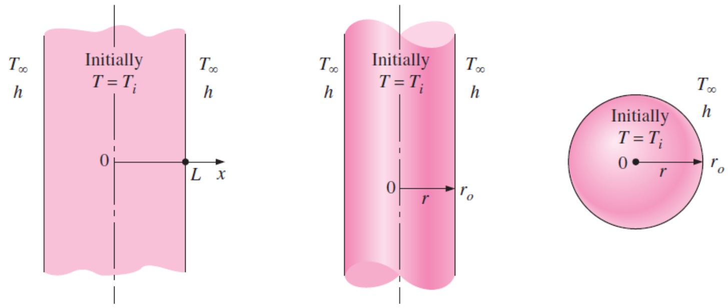
Physical situation	Nodal equation for $\Delta x = \Delta y$
(*a*) Interior node 	$[1+4(\text{Fo})]T_{m,n}^{p+1} - \text{Fo}\left(T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m+1,n}^{p+1} + T_{m,n-1}^{p+1}\right) - T_{m,n}^{p} = 0$
(*b*) Convection boundary node 	$[1+2(\text{Fo})(2+\text{Bi})]T_{m,n}^{p+1} - \text{Fo}\left[T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} + 2T_{m-1,n}^{p+1} + 2(\text{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$
(*c*) Exterior corner with convection boundary 	$[1+4(\text{Fo})(1+\text{Bi})]T_{m,n}^{p+1} - 2(\text{Fo})\left[T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1} + 2(\text{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$

Table 4-3 | (*Continued*).

Physical situation	Nodal equation for $\Delta x = \Delta y$
(*d*) Interior corner with convection boundary 	$\left[1+4(\mathrm{Fo})\left(1+\frac{\mathrm{Bi}}{3}\right)\right] T_{m,n}^{p+1} - \frac{2(\mathrm{Fo})}{3} \times \left[2T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1} + 2T_{m,n+1}^{p+1} + 2T_{m+1,n}^{p+1} + 2(\mathrm{Bi})T_{\infty}^{p+1}\right] - T_{m,n}^{p} = 0$
(*e*) Insulated boundary 	$[1+4(\mathrm{Fo})]T_{m,n}^{p+1} - \mathrm{Fo}\left(2T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}\right) - T_{m,n}^{p} = 0$

4.7 Transient heat conduction in large plane walls, long cylinders, and spheres with spatial effects

The temperature within a body will change from point to point as well as with time. In this section, we consider the variation of temperature with time and position in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.



(*a*) A large plane wall (*b*) A long cylinder (*c*) A sphere

Dimensionless temperature: $$\theta(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty}$$

Dimensionless distance from the center: $$X = \frac{x}{L}$$

Dimensionless heat transfer coefficient: $$\text{Bi} = \frac{hL}{k} \quad \textbf{(Biot number)}$$

Dimensionless time: $$\tau = \frac{\alpha t}{L^2} \quad \textbf{(Fourier number)}$$

Plane wall: $$\theta(x, t)_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta(r, t)_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2 \qquad (4.21)$$

Sphere: $$\theta(r, t)_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Center of plane wall ($x = 0$): $$\theta_{0,\text{ wall}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder ($r = 0$): $$\theta_{0,\text{ cyl}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \qquad (4.22)$$

Center of sphere ($r = 0$): $$\theta_{0,\text{ sph}} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

TABLE 4–1

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–2

The zeroth- and first-order Bessel functions of the first kind

ξ	$J_o(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

Where the first column is λ_1